Categorifying Type B Schur Algebras and Beyond ... (Cyclotomic)

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QUANTUM SCHUR-WEYL DUALITY

$q \in \mathbb{C}(q)$ generic , $V = \mathbb{C}(q)^n$

H_d^A : Type A Hecke alg / $\mathbb{C}(q)$ with

generators : $T_1, \dots, T_{d-1}$

relations $(T_i - 1)(T_i + q^{-1}) = 0$

$T_i T_j = T_j T_i \quad |i-j|>1$

$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad 1 \le i \le d-2$

$\curvearrowright V^{\otimes d}$ deformed permutations

($q \rightsquigarrow 1$: permutations)

$U_q(\mathfrak{gl}_n(\mathbb{C}))$: Type A quantum gp gen. by E_i, F_i $(1 \le i \le n-1)$
$K_i^{\pm 1}$ $(1 \le i \le n)$

subject to Serre relations &

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{q - q^{-1}}$$

$\vdots$

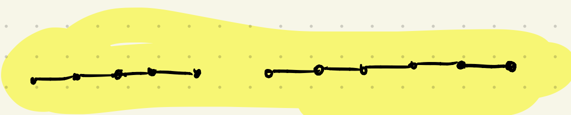
$$U_q(\mathfrak{gl}_n(\mathbb{C})) \curvearrowright V^{\otimes d}$$

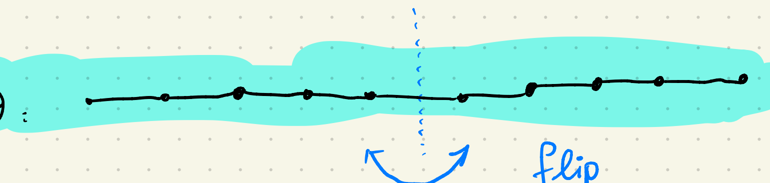
Th (Jimbo)

$$\boxed{U_q(\mathfrak{gl}_n(\mathbb{C})) \twoheadrightarrow \operatorname{End}_{H_d^A}(V^{\otimes d}) =: S_{n,d}^A \quad \text{Schur algebra}}$$

TYPE B

Type A	$\mathfrak{gl}_n$	Sym. Gp S_d
Type B	$\mathfrak{gl}_{n_1} \oplus \mathfrak{gl}_{n_2}$	$\mathbb{Z}_2 \wr S_d$
Type B	$\mathfrak{gl}_n^{\theta}$	$\mathbb{Z}_2 \wr S_d$



θ :  flip

Another "Type B"

(Ehrig-Stroppel) $\quad \mathfrak{osp}(n|m) \twoheadrightarrow \mathrm{End}_{Br_d(\delta)}(V^{\otimes d})$

$Br_d(\delta)$: Brauer Algebra $\rightsquigarrow$ Nazarov-Wenzl al

Type A : Hecke alg $\rightsquigarrow$ affine Hecke alg $\rightsquigarrow$ quiver Hecke alg

CLASSICAL CASE

Type B Weyl group B_d

gen. by $s_0, s_1, \cdots, s_{d-1}$ subject to Type A relations &

$s_0^2 = 1$ ($s_0^r = 1$ cyclotomic) $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$

Action by Signed Permutations (Green) Action 1

$V = \mathbb{C}\text{-}\langle v_1, \cdots, v_n, v_{-1}, \cdots, v_{-n}, v_0 \rangle$

optional

$s_1, \cdots, s_{d-1} \curvearrowright$ place permutation

$s_0 . v_{i_1} \otimes \cdots \otimes v_{i_d} = v_{-i_1} \otimes \cdots \otimes v_{i_d}$

Action 2

$$V = \mathbb{C}\text{-}\langle w_1, \cdots, w_{n_1}, w_{\bar{1}}, \cdots, w_{\bar{n}_2} \rangle$$

$$S_0 . v_{i_1} \otimes \cdots \otimes v_{i_d} = v_{i_1} \otimes \cdots \otimes v_{i_d} \quad \text{if } i_1 \in \{1, \cdots, n_1\}$$

$$S_0 . v_{i_1} \otimes \cdots \otimes v_{i_d} = -v_{i_1} \otimes \cdots \otimes v_{i_d} \quad \text{if } i_1 \in \{\bar{1}, \cdots, \bar{n}_2\}$$

Th (Mazorchuk-Stroppel, Hu-Stoll, Kujawa-Z.) $\quad U(\mathfrak{gl}_{n_1}) \otimes U(\mathfrak{gl}_{n_2}) \twoheadrightarrow \mathrm{End}_{\mathbb{C}B_d}\left((\mathbb{C}^{n_1+n_2})^{\otimes d} \right)$

(action 2)

Cyclotomic ✓

QUANTUM CASE

$q \in \mathbb{C}(q, Q_1, Q_2)$ generic,

H_d^B: Type B Hecke alg / $\mathbb{C}(q, Q_1, Q_2)$ with

generators: $T_0, T_1, \ldots, T_{d-1}$

relations $(T_i - 1)(T_i + q^{-1}) = 0$

$T_i T_j = T_j T_i \quad |i-j|>1 \qquad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \qquad 1 \leq i \leq d-2$

$(T_i - Q_1)(T_i - Q_2) = 0, \quad T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0$

$V = \mathbb{C}(q, Q_1, Q_2)^N, \qquad N = n_1 + n_2$

Th (Shoji-Sakamoto)

cyclotomic ✓

$$\boxed{U_q(\mathfrak{gl}_{n_1}) \otimes U_q(\mathfrak{gl}_{n_2}) \twoheadrightarrow \mathrm{End}_{H_d^B}(V^{\otimes d}) =: S^d_{n_1, n_2} \text{ Schur algebra of Type B}}$$

action 2

AMBIGUITY

① (Ariki) $\quad \mathrm{End}_{H_d^B}\Big(\bigoplus_{\substack{\mu \vdash d \\ \ell(\mu)\le n_1+n_2}} M(\mu)\Big)$ $\quad$ first kind $\quad$ cyclotomic ✓

② $S^d_{n_1,n_2} :\simeq \mathrm{End}_{H_d^B}\big((\mathbb{C}_q^{n_1+n_2})^{\otimes d}\big)$ $\quad$ second kind $\quad$ cyclotomic ✓

action 2

③ (Lai-Nakano-Xiang) $\quad \mathrm{End}_{H_d^B}\big((\mathbb{C}_q^{n_1+n_2})^{\otimes d}\big) =: S^d_{n_1,n_2}$

action 1

only in Type B

Relationships:

② special case of ① (Ariki)

① presentation by Wada

① $\cong$ ③ under separation conditions

$q^k Q_1 - Q_2$: invertible
$|k| < n_1 + n_2$

type B, classical case $\rightsquigarrow$ char $\Bbbk \neq 2$

Lai-Nakano-Xiang

$$S^B_{n_1,n_2,d} \cong \bigoplus_{d_1+d_2=d} S^A_{n_1,d_1} \otimes S^A_{n_2,d_2}$$

Ariki

$$S^B_{n_1,n_2,d} \cong \bigoplus_{d_1+d_2=d} S^A_{n_1,d_1} \otimes S^A_{n_2,d_2}$$

Consequence

$$S^B_{n_1,n_2,d} \cong S^B_{n_1,n_2,d}$$ Not explicit!

(Li-Z.) explicit isomorphism only in Type B

PRESENTING TYPE A SCHUR ALGEBRA

Lusztig's Quantum Group : $\dot{U}(gl_n)$

- generators $e_i, f_i, (1 \leq i \leq n-1)$, 1_λ for $\lambda \in \mathbb{Z}^n$,
- relations :

$$1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\lambda$$

$$e_i 1_\lambda = 1_{\lambda+\alpha_i} e_i = 1_{\lambda+\alpha_i} e_i 1_\lambda$$

$$f_i 1_\lambda = 1_{\lambda-\alpha_i} f_i \qquad (\alpha_i : \text{root for } e_i)$$

Th (Doty - Giaquito)

$$\boxed{\begin{array}{c} S^A_{n,d} \simeq \dot{U}(gl_n)/\sim \\ \sim : \ 1_\lambda = 0 \quad \forall \lambda \notin \Lambda = \{\lambda \in \mathbb{Z}^n \mid \lambda_i \geq 0, \ \Sigma \lambda_i = d\} \end{array}}$$

TYPE B PRESETATIONS

$$S^B_{n_1,n_2,d} \simeq \mathrm{End}_{H^B_d}\left(\left(\mathbb{C}_q^{n_1+n_2}\right)^{\otimes d}\right)$$

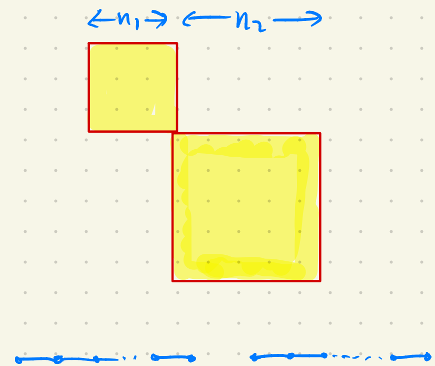
$N = n_1 + n_2$

Action 2

Th (Kujawa-Z.)

the second kind

$$\boxed{\begin{gathered} S^B_{n_1,n_2,d} \simeq \dot{U}(\mathfrak{gl}_{n_1} \oplus \mathfrak{gl}_{n_2}) / \sim \\ \sim :\ 1_\lambda = 0 \\ \forall \lambda \notin \Lambda = \{\lambda \in \mathbb{Z}^N \mid \lambda_i \geq 0,\ \Sigma \lambda_i = d\} \end{gathered}}$$



cyclotomic ✓

(Wada) Presentation of cyclotomic q-Schur alg of the first kind.

ALTERNATIVE CONSTRUCTION

(Bao - Kujawa - Wang - Li) $(D = 2d,\ n_1 = n_2 = n,\ N = 2n)$

$\mathcal{X} = \{ \{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_N \subseteq \mathbb{F}_q^D \}$ N - step flags

isotropic : $V_i^\perp = V_{N-i}$ w.r.t sym. bil. form

$\mathcal{U}^\imath \simeq$ orbit functions on $\mathcal{X} \times \mathcal{X} / O_N$

$m : \mathcal{U}^\imath \otimes \mathcal{U}^\imath \longrightarrow \mathcal{U}^\imath$ convolution

$\mathcal{U}^\imath$ has a presentation from Quantum Symmetric Pairs

gen. by $\mathbf{e_i}, \mathbf{f_i}, \mathbf{d_i}^{\pm 1}$

$\theta: \quad U(\mathfrak{gl}_N) \to U(\mathfrak{gl}_N)$

$$\boxed{\begin{aligned} I_\bullet &= \phi \\ I_\circ &= A_{N-1} \\ \theta &= \text{flip} \end{aligned}}$$

θ graph automorphism.

$\mathcal{U}(\mathfrak{gl}_N)^\theta \hookrightarrow \mathcal{U}(\mathfrak{gl}_N)$ subalg fixed by θ

(Letzter - Kolb) $\boxed{\text{presentation for } \mathcal{U}(\mathfrak{gl}_N)^\theta \xrightsquigarrow[\text{quantized}]{} \mathcal{U}^\iota}$

$\mathcal{U}^\iota \subseteq \mathcal{U}_q(\mathfrak{gl}_{2n}) \quad (m, i, \Delta, \varepsilon, S)$

$\Delta(\mathcal{U}^\iota) \not\subseteq \mathcal{U}^\iota \otimes \mathcal{U}^\iota$, but $\Delta(\mathcal{U}^\iota) \subseteq \mathcal{U}^\iota \otimes \mathcal{U}_q(\mathfrak{gl}_n)$

$\mathcal{U}^\iota$ coideal in $\mathcal{U}_q(\mathfrak{gl}_n)$

$\mathcal{Y} = \{ V_0 \subseteq V_1 \subseteq \cdots \subseteq V_D \subseteq \mathbb{F}_q^D \}$ $(D = 2d)$

$V = \mathbb{Q}(\sqrt{q})^{\otimes d} \simeq$ orbit function on $\mathcal{X} \times \mathcal{Y} / \Theta_N$

$H_d^B \simeq$ orbit function on $\mathcal{Y} \times \mathcal{Y} / \Theta_N$

$\mathcal{U}^\imath \curvearrowright V^{\otimes d}$, $H_d^B \curvearrowright V^{\otimes d}$ convolution

Th (Bao-Kujawa-Li-Wang)

Quantum

$$\boxed{\mathcal{U}^\imath \twoheadrightarrow \mathrm{End}_{H_d^B}(V^{\otimes d})}$$

RELATIONSHIP

Notice

$S_0 . (v_1 + v_{-1}) = v_1 + v_{-1}$ $\quad$ $S_0 . w_1 = w_1$

$S_0 . (v_1 - v_{-1}) = -(v_1 - v_{-1})$ $\quad$ $S_0 . w_{\bar{1}} = - w_{\bar{1}}$

Th (Li-Z) $\boxed{\exists \text{ base change } V^{\otimes d} \to V^{\otimes d} \quad (2 : \text{invertible})}$

$$\begin{tikzcd}
\mathcal{U}(\mathfrak{gl}_n \oplus \mathfrak{gl}_n) \arrow[r, two heads] \arrow[d, "\simeq"'] & \mathrm{End}_{\mathbb{C}B_d}(V^{\otimes d}) \arrow[d, "\simeq"] \\
\mathcal{U}(\mathfrak{gl}_{2n})^{\theta} \arrow[r, two heads] & \mathrm{End}_{\mathbb{C}B_d}(V^{\otimes d})
\end{tikzcd}$$

action 2

action 1

ANOTHER PRESENTATION

Th (Li-Z.)

$$\boxed{\begin{array}{l} S^B_{n,n,d} \simeq U(\mathfrak{gl}_N)^\theta / \sim \qquad \text{(classical)} \\ \quad \sim \text{ further relations} \end{array}}$$

E.g. extra relation

$n=2$

$d=3$

$$([\mathbf{e},\mathbf{f}]+\mathbf{d}_1-0)([\mathbf{e},\mathbf{f}]+\mathbf{d}_1-2)\cdots([\mathbf{e},\mathbf{f}]+\mathbf{d}_1-6)=0$$

(Bao-Kujawa-Li-Wang)

$$\boxed{\text{Conjectured full list of relations for } S^B_{n,n,d} \qquad \text{(quantum)}}$$

FUTURE WORK

- homomorphism / isomorphism in quantum setting

$$U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_n) \not\to \qquad U^\iota$$

Not a Hopf alg

- Complicated Relations

  Geometric proof in quantum setting

- Geometric interpretation in cyclotomic case

CATOGORIFICATION IN TYPE A

$\mathcal{U}$ graded 2-category (Khovanov-Lauda-Rouquier)

objects $\lambda \in \mathbb{Z}^n$

1-mor: $1_\lambda \in \mathrm{Hom}(\lambda, \lambda)$ identity morphism

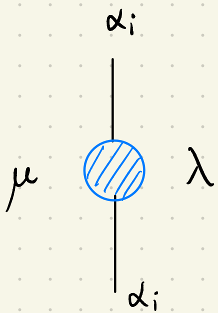
$\mathrm{Hom}(\lambda, \mu) = \mathbb{Z}$-span of words

$1_\mu e_{\pm i_1} \cdots e_{\pm i_s} 1_\lambda$ $(e_{-i} = f_i)$

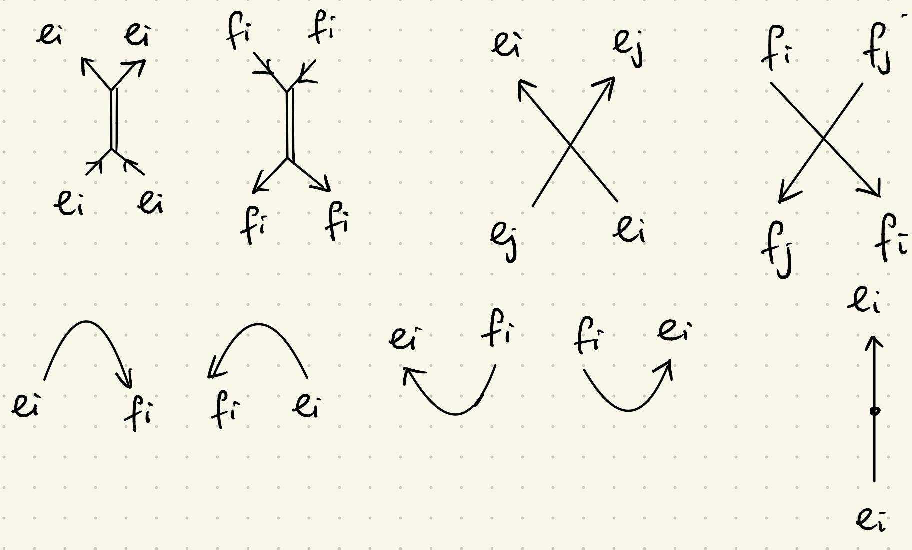
$\mu - \lambda = \Sigma \pm$ roots

e.g. $1_{\alpha_i + \lambda} e_i 1_\lambda \in \mathrm{Hom}(\lambda, \lambda + \alpha_i)$

2- morphisms



Vertical & horizontal stacking of



QUANTUM GROUPS

$\mathcal{C} = \mathrm{Kar}(\mathcal{U}) \supseteq \mathcal{U}$ image of idempotent exists

$K_0(\mathcal{C}) = \mathbb{Z}[q, q^{-1}] \langle \text{isoclasses of 1-mor} \rangle / \sim$.

$q \curvearrowright K_0(\mathcal{C})$ grade shift

$K_0(\mathcal{C}) \otimes K_0(\mathcal{C}) \longrightarrow K_0(\mathcal{C})$: composition of 1-mor.

Th (Khovanov-Lauda, Rouquier)

$$\boxed{K_0(\mathrm{Kar}(\mathcal{U})) \simeq U_q(\mathfrak{sl}_n) \quad \text{as } \mathbb{Z}[q, q^{-1}]\text{-alg}}$$

"Schur Category" . $S \simeq \dot{\mathcal{U}} / \sim$ $\qquad \lambda = 0 \quad \lambda \notin \Lambda$

Th (Mackaay – Stosic – Vaz)

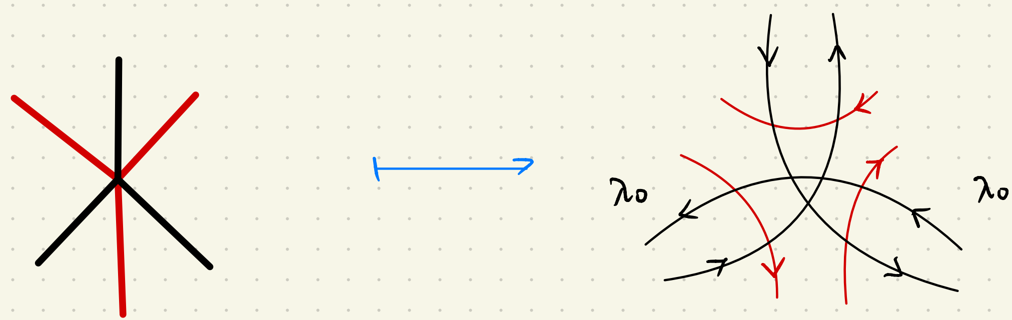
$$\boxed{K_0(\mathrm{Kar}(S)) \simeq S^A_{n,d}}$$

alg ver.

$$\boxed{H^A_d \simeq 1_{\lambda_0} S^A_{n,d} 1_{\lambda_0}}$$

Proof / byproduct

Hecke 2-category $\hookrightarrow S^A_{n,d}$



$\lambda_0 = (1^d)$

TYPE B SCHUR ALGEBRAS

Work-in-Progress (Kujawa - Z.)

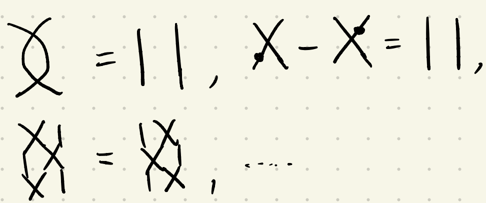
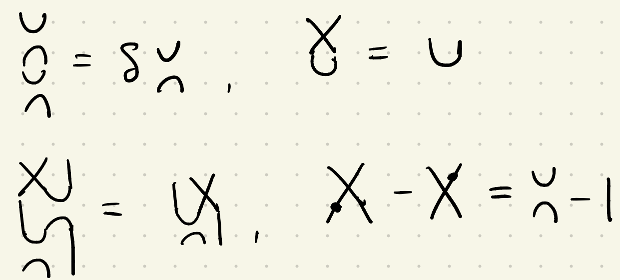
$\Gamma = A_{n_1} \cup A_{n_2} =$ 

$S(\Gamma) := U(\Gamma)/\sim$

$\sim$: λ if $\lambda \notin \Lambda = \{\lambda \in \mathbb{Z}_{\geq 0}^{n_1+n_2} \mid \sum \lambda_i = d\}$

Goal: $K_0(\mathrm{Kar}(S(\Gamma))) \cong S^B_{n_1,n_2,d}$

(Also: cyclotomic case)

	Type A ($U_q(\mathfrak{sl}_n)$)	Type B ($\mathcal{U}^\iota$) Ehrig - Stroppel
Weak categorification 1-morphism functors	module category for cyclotomic quotients of AHA. e_i, f_i : functors via restriction / induction	parabolic category $\mathcal{O}$ for O_N e_i, f_i : translation functors via $pr \circ (- \otimes V)$
Strong categorification 2-morphisms natural transformations	KLR - alg $x_i =$  $\tau_i =$  	Nazorov - Wenzl alg  .....

FUTURE WORK

- diagrammatically categorify $\mathcal{U}^{\iota}$

- Soergel calculus $\hookrightarrow$ $\mathcal{S}^{\iota} := \mathcal{U}^{\iota}/\sim$