Two-boundary degenerate affine Hecke algebras for The Lie superalgebras $\mathfrak{gl}(n|m)$ and $\mathfrak{q}(n)$

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History

$$A \curvearrowright W \curvearrowleft B$$

	A	W	B
Schur-Weyl		$V^{\otimes d}$	S_d
Arakawa-Suzuki	$\mathfrak{sl}_n(\mathbb{C})$	one bndry $M \otimes V^{\otimes d}$	$\mathbb{C}[x_1, \dots, x_d] \otimes S_d$
Daugherty		two bndry $M \otimes N \otimes V^{\otimes d}$	deg. 2-boundary affine Hecke alg
Sergeev		$V^{\otimes d}$	$S_d \otimes Cl_d$
Hill-Kujawa-Sussan	$\mathfrak{q}_n(\mathbb{C})$	$M \otimes V^{\otimes d}$	$\mathbb{C}[x_1, \dots, x_d] \otimes S_d \otimes Cl_d$
Z.		$M \otimes N \otimes V^{\otimes d}$	?

Lie Superalgebra of Type Q

$$\mathfrak{g} = \mathfrak{q}_n(\mathbb{C}) = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$$

$$\mathfrak{g}_{\bar{0}} = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \,\middle|\, A \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \right\} \qquad \text{even}$$

$$\mathfrak{g}_{\bar{1}} = \left\{ \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \,\middle|\, B \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \right\} \qquad \text{odd}$$

with Lie super bracket

$$[x, y] = xy - (-1)^{\bar{x}\bar{y}}\, yx$$

x: homogeneous of deg $\bar{x}$

y: homogeneous of deg $\bar{y}$

$$V = \mathbb{C}^{\overset{\text{even}}{n}|\underset{\text{odd}}{n}}$$

Degenerate 2-boundary affine Hecke-Clifford Superalgebra

- Goal : M, N : $\mathfrak{q}(n)$-mods

$$\mathfrak{q}(n) \curvearrowright M \otimes N \otimes V^{\otimes d} \curvearrowleft ?$$

$\mathcal{H}_d$: Superalg. gen. by

odd	$x_1, \dots, x_d$	$\rightsquigarrow$ Polynomial Relations
odd	$y_1, \dots, y_d$	
odd	Z_0	
odd	$c_1, \dots, c_d$	$\rightsquigarrow$ Clifford relations
even	$s_1, \dots, s_{d-1}$	$\rightsquigarrow$ Symmetric Group Relations

And Mixed Relations such as ...

$$s_i x_i = x_{i+1} s_i + (c_i - c_{i+1})$$

$$z_0 (x_1 + y_1) = -(x_1 + y_1) z_0$$

and more ....

- Goal: Defined an action $\mathcal{H}_d \curvearrowright M \otimes N \otimes V^{\otimes d}$

  - Symmetric Group $\curvearrowright$ via Signed Permutations

    i.e. $v \otimes w \mapsto (-1)^{\bar{v}\bar{w}} w \otimes v$

– Clifford Generators :

$$C \in \mathrm{End}_{q(n)}(V), \qquad C = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \qquad c^2 = -\mathrm{id}$$

odd

Note V : Simple,

Yet $\quad \mathrm{End}_{q(n)}(V) = \langle \mathrm{id}_V , C \rangle$

" Type Q Schur's Lemma "

$$C_i : \quad M \otimes N \otimes V \otimes \cdots \otimes \underset{\substack{\circlearrowright \\ C}}{\overset{i}{V}} \otimes \cdots \otimes V$$

Tensor Casimir Element

$$e_i = \begin{bmatrix} E_{ij} & 0 \\ 0 & E_{ij} \end{bmatrix} \qquad f_{ij} = \begin{bmatrix} 0 & E_{ij} \\ E_{ij} & 0 \end{bmatrix}$$

$$\Omega = \sum e_{ij} \otimes f_{ji} - \sum f_{ij} \otimes e_{ji} \in q(n) \otimes q(n)$$

$$\Omega \, \Delta(x) = (-1)^{\bar{x}\bar{\Omega}} \Delta(x) \, \Omega \qquad \forall x \in q(n)$$

$\Longrightarrow$ view Ω in $\mathrm{End}_{q(n)}(W_1 \otimes W_2)$ i.e. acts on $W_1 \otimes W_2$

Applications: Translation functors

Action of The Polynomial Generators

x_1 : $M \otimes N \otimes V \otimes \cdots$

Ω

y_1 : $M \otimes N \otimes V \otimes \cdots$

Ω

z_0 : $M \otimes N \otimes V \otimes \cdots$

Ω

Moreover,

x_i : $(M \otimes V^{\otimes i-1}) \otimes V \otimes V^{\otimes d-i} \otimes N$

Ω

IS

$M \otimes N \otimes V^{\otimes d}$

JM elt: $V^{\otimes i-1} \otimes V$

Ω

$\Omega \circ (1 \otimes C) =$ signed swap in $\mathrm{End}_{q(n)}(V \otimes V)$

Main Result

Upshot

$$\Omega \curvearrowright \quad M \otimes N \otimes V^{\otimes d}$$

$\rightsquigarrow$ action of x_i, y_i, z_0, s_i

$$c_i \qquad M \otimes N \otimes V \otimes V \otimes \cdots \quad \otimes V$$

Th (2.) $\quad \mathcal{H}_d \longrightarrow \mathrm{End}_{q(n)}(M \otimes N \otimes V^{\otimes d})$

i.e. all relations are verified

Classifying Irred. Modules

for nice M, N, x_1, y_1 acts by eigenvalues

$\sim$: relations for x_1, y_1 (dependent on M, N)

- Goal: construct & Classify $\widetilde{\mathcal{H}}_d := \mathcal{H}_d / \sim$

  all irred. calibrated representations of $\widetilde{\mathcal{H}}_d$

- Tool: Decomposing $M \otimes N \otimes V^{\otimes d}$

Calibrated Modules

Def. A Calibrated module for $\widetilde{\mathcal{H}}_d$ $\Leftrightarrow$ Z_i acts semisimply $\forall i$

$Z_i = x_i + y_i$ — ith JM elt

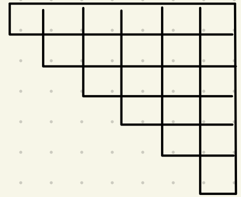
$$Z_i : \quad (M \otimes N \otimes V^{\otimes i-1}) \otimes V$$

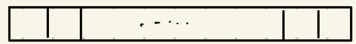
$$\Omega$$

Idea: $\langle Z_i \rangle$: abelian subalg $\subseteq \widetilde{\mathcal{H}}_d$

Compare : $\mathfrak{h} \subseteq \mathfrak{g}$ abelian Lie subalg

Combinatorics

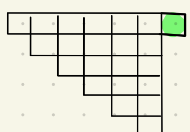
M: highest weight  $= (n, n-1, n-2, \cdots, 2, 1)$

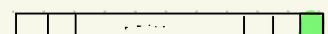
N: highest weight  $= (\ell, 0, 0, \cdots, 0)$

V: highest weight  $= (1, 0, 0, \cdots, 0)$

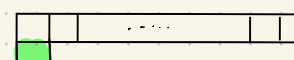
NICE in the sense:

$M \otimes N$: "multiplicity free".

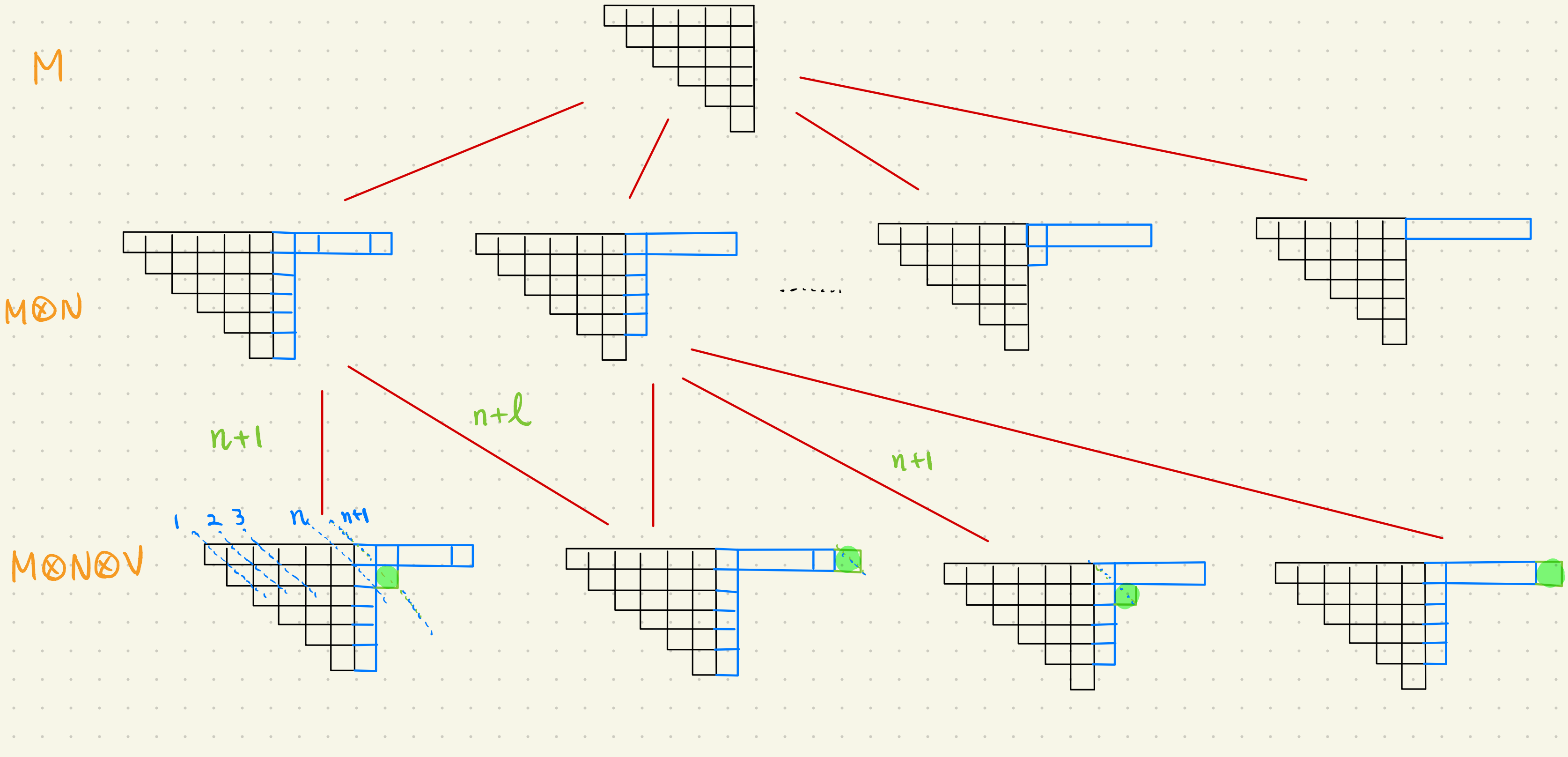
$M \otimes V \simeq L($  $)^{\oplus 2}$,

$N \otimes V \simeq L($  $)^{\oplus 2}$

$\oplus$

$L($  $)^{\oplus 2}$

Decomposing $M \otimes N \otimes V^{\otimes d}$



Calibrated Modules D^λ

λ: portion in row d

Clifford gen. by c_i

$$D^\lambda = \langle x v_T \mid x \in Cl_d ,\ T: \text{path} \rangle$$

Paths in the diagram to λ $\rightsquigarrow$ Basis elt (over Cl_d) for D^λ

Labels on the edges $\rightsquigarrow$ eigenvalues for z_i - action

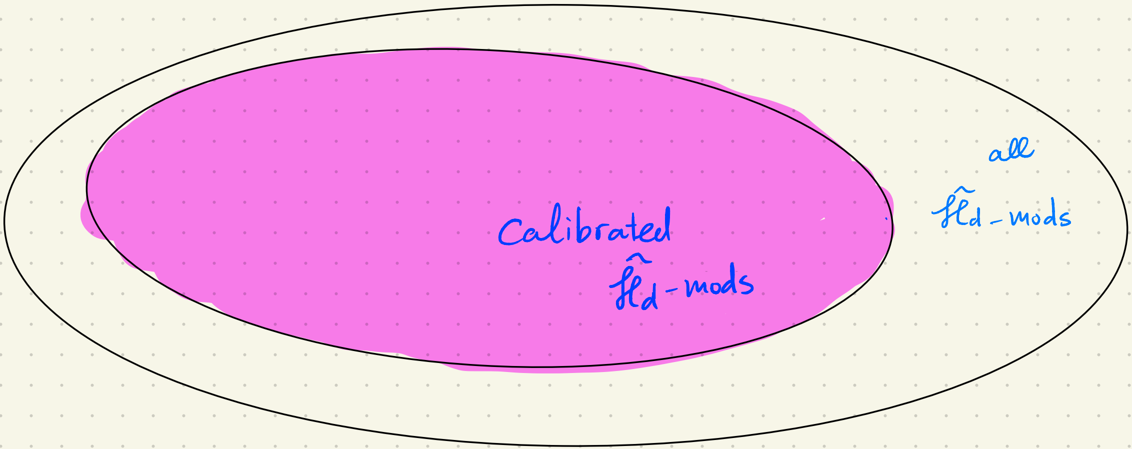
s_i $\rightsquigarrow$ "permuting paths"

c_i $\rightsquigarrow$ left multiplication in Cl_d

Main Result

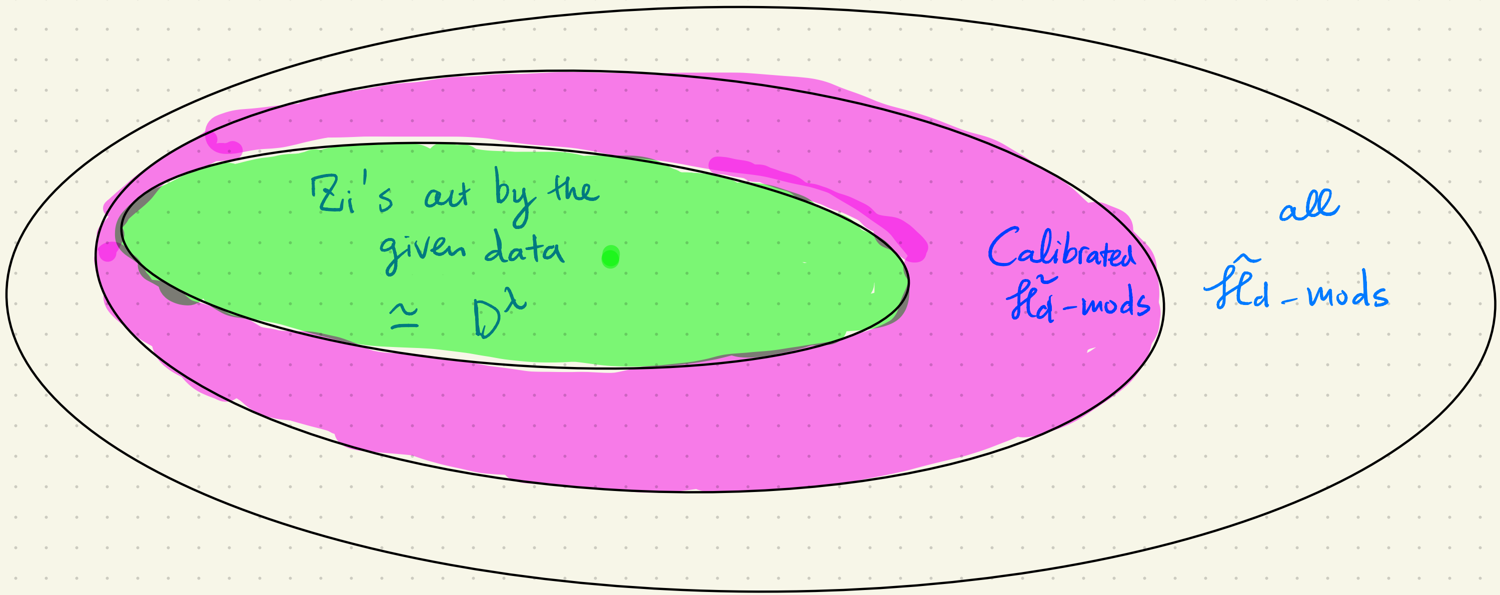
Th (2.) D^λ is Irreducible as an $\tilde{\mathcal{H}}_d$ ($\mathcal{H}_d$) -mod.

Th (2.) All irred. calibrated representations are uniquely determined by eigenvalues of Z_i



Th(2.) If Z_i's act according to the data $\Rightarrow D^\lambda$

Irred. $\tilde{\mathcal{H}}_d$ - summands in $M \otimes N \otimes V^{\otimes d} \simeq D^\lambda$ $\exists \lambda$ in d^{th} Row

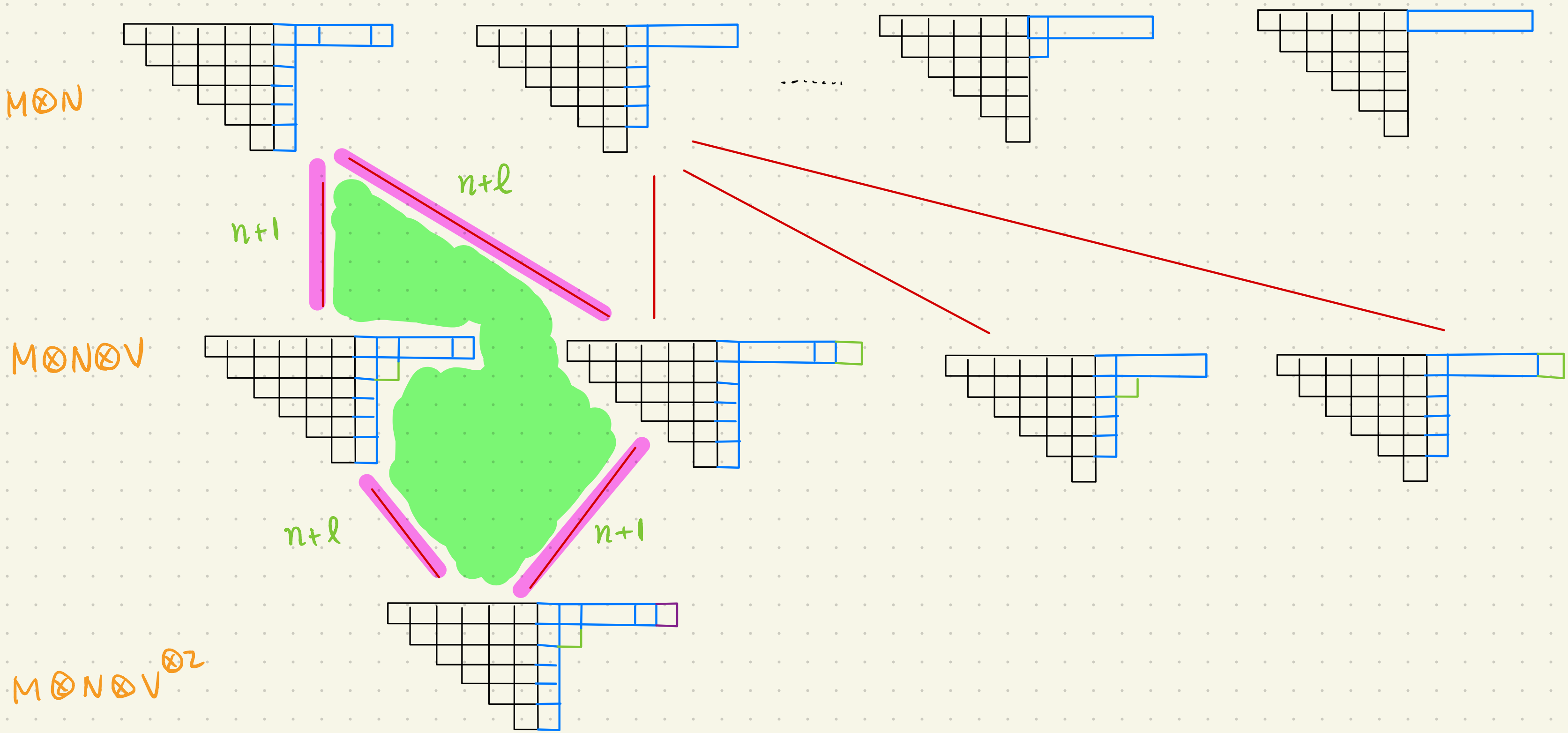


Other Cases

- $\mathfrak{gl}(n|m)$: similar story
  "No Clifford"
  Use Hecke alg from Daugherty
- $\mathfrak{p}(n)$ BDEHHILNSS "WINART project"
  Casimir elt ?
  M, N: Verma modules ?

THANK YOU !

Action of S_1



$M \otimes N$
$M \otimes N \otimes V$
$M \otimes N \otimes V^{\otimes 2}$
$n+1$
$n+l$
$n+l$
$n+1$