

Two-boundary degenerate affine Hecke algebras for

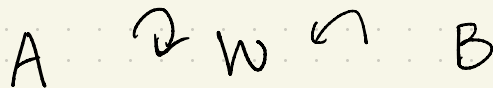
The Lie superalgebras $gl(n|m)$ and $o(n)$

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History



	A	W	B
Schur-Weyl	$\mathfrak{sl}_n(\mathbb{C})$	$V^{\otimes d}$	S_d
Arakawa-Suzuki		one bndry $M \otimes V^{\otimes d}$	$\mathbb{C}[x_1, \dots, x_d] \otimes S_d$
Daugherty		two bndry $M \otimes N \otimes V^{\otimes d}$	deg. 2-boundary affine Hecke alg
Sergeev	$\mathfrak{q}_n(\mathbb{C})$	$V^{\otimes d}$	$S_d \otimes \mathbb{C}l_d$
Hill-Kujawa-Sussan		$M \otimes V^{\otimes d}$	$\mathbb{C}[x_1, \dots, x_d] \otimes S_d \otimes \mathbb{C}l_d$
Z.		$M \otimes N \otimes V^{\otimes d}$?

Lie Superalgebra of Type Q

$$\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$$

$$\mathfrak{g}_{\bar{0}} = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \mid A \in \text{Mat}_{n \times n}(\mathbb{C}) \right\} \quad \text{even}$$

$$\mathfrak{g}_{\bar{1}} = \left\{ \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \mid B \in \text{Mat}_{n \times n}(\mathbb{C}) \right\} \quad \text{odd}$$

with Lie super bracket

$$[x, y] = xy - (-1)^{\bar{x}\bar{y}} yx$$

x : homogeneous of deg \bar{x}

y : homogeneous of deg \bar{y}

$$V = \mathbb{C}^{n|n} \begin{matrix} \swarrow \text{even} \\ \searrow \text{odd} \end{matrix}$$

Degenerate 2-boundary affine Hecke-Clifford Superalgebra

● Goal: M, N : $\mathfrak{g}(\mathfrak{m})$ -mods

$$\mathfrak{g}(\mathfrak{m}) \rightsquigarrow M \otimes N \otimes V^{\otimes d} \leftarrow ?$$

\mathcal{H}_d : Superalg. gen. by

odd x_1, \dots, x_d
odd y_1, \dots, y_d \rightsquigarrow Polynomial Relations

odd \mathbb{Z}_0

odd c_1, \dots, c_d \rightsquigarrow Clifford relations

even s_1, \dots, s_{d-1} \rightsquigarrow Symmetric Group Relations

And Mixed Relations such as ...

$$S_i x_i = x_{i+1} S_i + (C_i - C_{i+1})$$

$$Z_0(x_i + y_i) = -(x_i + y_i) Z_0$$

and more ...

● Goal: Defined an action $\mathfrak{sl}_d \curvearrowright M \otimes N \otimes V^{\otimes d}$

● - Symmetric Group \curvearrowright via Signed Permutations

i.e. $v \otimes w \mapsto (-1)^{\overline{v \overline{w}}} w \otimes v$

- Clifford Generators :

$$C \in \text{End}_{q(n)}(V), \quad C = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \quad C^2 = -\text{id}$$

odd

Note V : Simple,

$$\text{Yet } \text{End}_{q(n)}(V) = \langle \text{id}_V, C \rangle$$

"Type Q Schur's Lemma"

$$C_i : \quad M \otimes N \otimes V \otimes \dots \otimes \overset{i}{V} \otimes \dots \otimes V$$

\uparrow
 C

Tensor Casimir Element

$$e_i = \begin{bmatrix} E_{ij} & 0 \\ 0 & E_{ij} \end{bmatrix} \quad f_{ij} = \begin{bmatrix} 0 & E_{ij} \\ E_{ij} & 0 \end{bmatrix}$$

$$\Omega = \sum e_{ij} \otimes f_{ji} - \sum f_{ij} \otimes e_{ji} \in \mathfrak{g}(\mathfrak{n}) \otimes \mathfrak{g}(\mathfrak{n})$$

$$\Omega \Delta(x) = (-1)^{\bar{x}\bar{\Omega}} \Delta(x) \Omega \quad \forall x \in \mathfrak{g}(\mathfrak{n})$$

\Rightarrow view Ω in $\text{End}_{\mathfrak{g}(\mathfrak{n})}(W_1 \otimes W_2)$ i.e. acts on $W_1 \otimes W_2$

Applications: Translation functors

Action of The Polynomial Generators

$x_1:$ $M \otimes N \otimes V \otimes \dots$
 $\uparrow \quad \Omega \quad \uparrow$

$y_1:$ $M \otimes N \otimes V \otimes \dots$
 $\uparrow \quad \Omega \quad \uparrow$

$z_0:$ $M \otimes N \otimes V \otimes \dots$
 $\uparrow \quad \Omega \quad \uparrow$

Moreover,

$x_i:$ $(M \otimes V^{\otimes i-1}) \otimes V \otimes V^{\otimes d-i} \otimes N$
 $\uparrow \quad \Omega \quad \uparrow$ IS
 $M \otimes N \otimes V^{\otimes d}$

JM elt:

$V^{\otimes i-1} \otimes V$
 $\uparrow \quad \Omega \quad \uparrow$

$\Omega \circ (1 \otimes C) = \text{signed swap in } \text{End}_{q(n)}(V \otimes V)$

Main Result

Upshot

$$\Omega \leadsto M \otimes N \otimes V^{\otimes d}$$

\rightsquigarrow action of x_i, y_i, z_0, s_i

$$C_i \quad \begin{array}{c} \curvearrowright \quad \curvearrowright \\ M \otimes N \otimes V \otimes V \otimes \dots \end{array} \quad \begin{array}{c} \curvearrowright \\ \otimes V \end{array}$$

Th (2.) $\mathfrak{h}_d \rightarrow \text{End}_{\text{qm}}(M \otimes N \otimes V^{\otimes d})$

i.e. all relations are verified

Classifying Irred. Modules

for nice M, N , x_1, y_1 acts by **eigenvalues**

\sim : relations for x_1, y_1 (dependent on M, N)

● Goal: Construct & Classify $\tilde{\mathfrak{sl}}_d := \mathfrak{sl}_d / \sim$

all irred. **calibrated** representations of $\tilde{\mathfrak{sl}}_d$

● Tool: Decomposing $M \otimes N \otimes V^{\otimes d}$

Calibrated Modules

Def. A Calibrated module for $\tilde{\mathfrak{h}}_d \Leftrightarrow \mathbb{Z}_i$ acts semi simply $\forall i$

$\mathbb{Z}_i = x_i + y_i - i$ th JM elt

$$\mathbb{Z}_i: (M \otimes N \otimes V^{\otimes i-1}) \otimes V$$

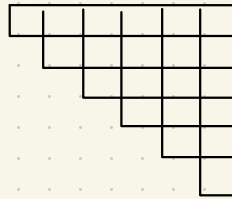
$\nwarrow \quad \nearrow$
 Ω

Idea: $\langle \mathbb{Z}_i \rangle$: abelian subalg $\subseteq \tilde{\mathfrak{h}}_d$

Compare: $\mathfrak{h} \subseteq \mathfrak{g}$ abelian Lie subalg

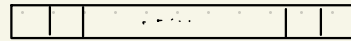
Combinatorics

M: highest weight



$$= (n, n-1, n-2, \dots, 2, 1)$$

N: highest weight



$$= (l, 0, 0, \dots, 0)$$

V: highest weight



$$= (1, 0, 0, \dots, 0)$$

NICE in the sense:

$M \otimes N$: "multiplicity free".

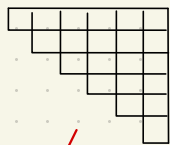
$$M \otimes V \cong L \left(\begin{array}{cccccc} \square & & & & & \square \\ \square & \square & & & & \\ \square & \square & \square & & & \\ \square & \square & \square & \square & & \\ \square & \square & \square & \square & \square & \\ \square & \square & \square & \square & \square & \square \end{array} \right)^{\oplus 2}$$

$$N \otimes V \cong L \left(\begin{array}{cccccc} \square & \square & \dots & \square & \square & \square \end{array} \right)^{\oplus 2}$$

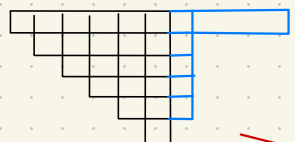
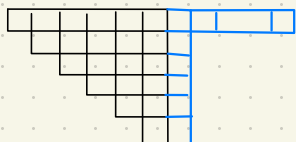
$$L \left(\begin{array}{cccccc} \square & \square & \dots & \square & \square & \square \\ \square & & & & & \square \end{array} \right)^{\oplus 2}$$

Decomposing $M \otimes N \otimes V^{\otimes d}$

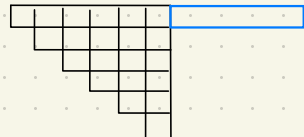
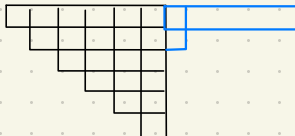
M



$M \otimes N$



...

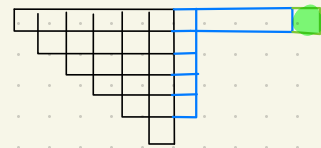
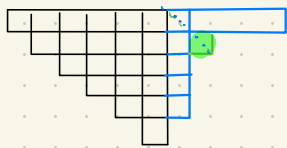
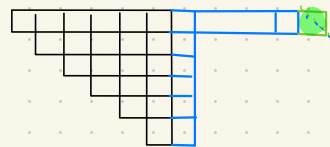
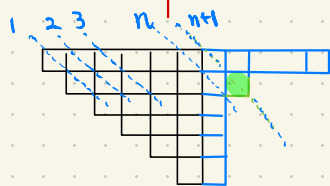


$n+1$

$n+l$

$n+1$

$M \otimes N \otimes V$



...

Calibrated Modules D^λ

λ : partition in row d

Clifford gen. by c_i

$$D^\lambda = \langle x v_T \mid x \in Cl_d, T: \text{path} \rangle$$

Paths in the diagram to λ \rightsquigarrow Basis elt (over Cl_d) for D^λ

Labels on the edges \rightsquigarrow eigenvalues for Z_i -action

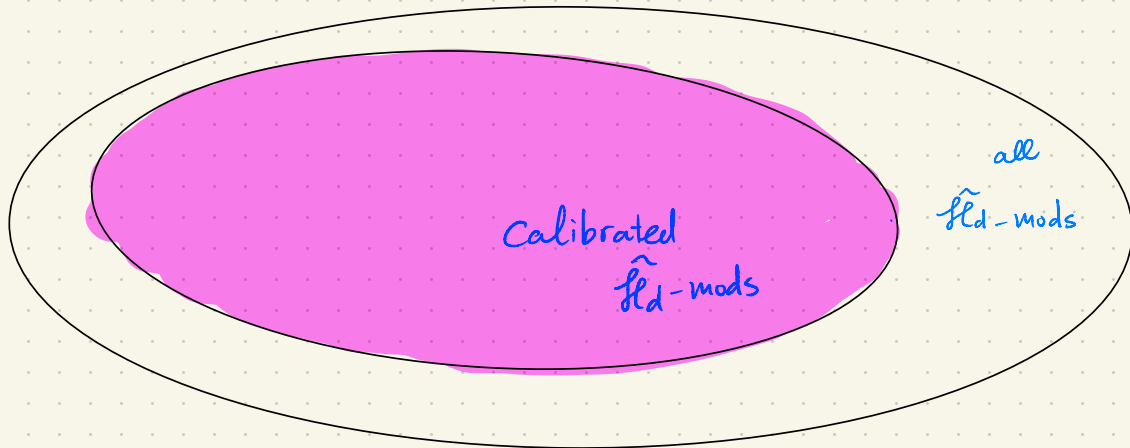
S_i \rightsquigarrow "permuting paths"

c_i \rightsquigarrow left multiplication in Cl_d

Main Result

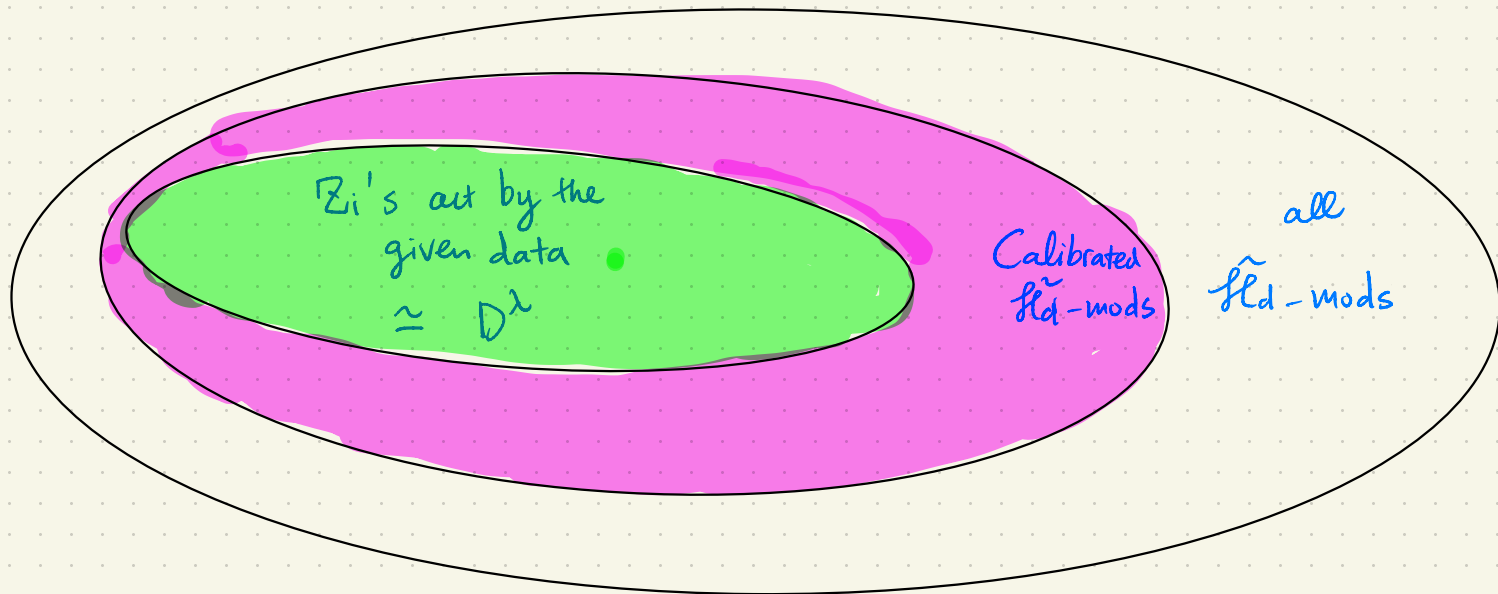
Th (2.) D^λ is Irreducible as an $\tilde{\mathfrak{h}}_d(\mathfrak{h}_d)$ -mod.

Th (2.) All irred. calibrated representations are uniquely determined by eigenvalues of Z_i



Th(z.) If Z_i 's act according to the data $\Rightarrow D^\lambda$

Irred. $\tilde{\mathcal{H}}_d$ - summands in $M \otimes N \otimes V^{\otimes d} \simeq D^\lambda \ni \lambda$ in d^{th} Row



Other Cases

- $gl(n/m)$ similar story
- "No Clifford"
Use Hecke alg from Daugherty
- $p(n)$ BDEHHILNSS "WINART project"
Casimir elt?
M, N: Verma modules?

THANK YOU!

Action of S_1

