M Diagram Basis of The Specht Module for (n,n,n)

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Outline of the Talk

- 3 constructions of Specht module
- Motivations / future work

1. Polytabloid Basis

char $\Bbbk = 0$ $\qquad$ S_d: symmetric group

$$\{\text{irred reps of } S_d\}$$

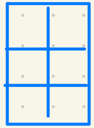
$$\updownarrow$$

$$\text{conjugacy classes of } S_d$$

$$\updownarrow$$

$$\text{cycle types, i.e. Partitions}$$

e.g. $(2,2,2)$: partition of 6



Young diagram

D^λ : irred. mod for partition λ

Th (Specht, Young, etc)

D^λ has a $\Bbbk$-basis

$$\{ v_T \mid T : \text{standard Young tableaux of shape } \lambda \}$$

e.g. $\lambda = (2,2,2)$

$$\underset{T_0}{\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline\end{array}} \quad \underset{T_1}{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline\end{array}} \quad \underset{T_2}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline\end{array}} \quad \underset{T_3}{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline\end{array}} \quad \underset{T_4}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline\end{array}}$$

Partial order : left weak Bruhat order

$$T_i = T_0 . w_i$$

$$\begin{tikzcd}
 & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline\end{array} \; T_4 = T_0 . s_3 s_2 s_4 \arrow[dl, dash] \arrow[dr, dash] & \\
T_2 = T_0 . s_3 s_2 \; \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline\end{array} \arrow[dr, dash] & & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline\end{array} \; T_3 = T_0 . s_3 s_4 \arrow[dl, dash] \\
 & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline\end{array} \; T_1 = T_0 . s_3 \arrow[d, dash] & \\
 & \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline\end{array} \; T_0 = T_0 . \mathrm{id} &
\end{tikzcd}$$

2. Web Basis

$V = \Bbbk^3$: natural module for $\mathfrak{sl}_3(\Bbbk)$

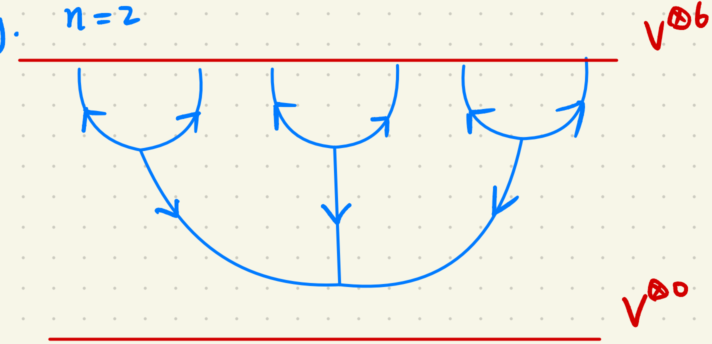
$V^{\otimes 3n}$ $(\mathfrak{sl}_3, S_{3n})$-bimod

$\mathrm{Hom}_{\mathfrak{sl}_3}(\Bbbk, V^{\otimes 3n})$, mod-$S_{3n}$

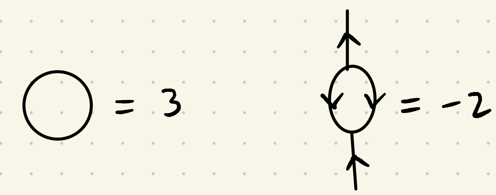
Th. (Kuperberg, Kuperberg-Khovanov)

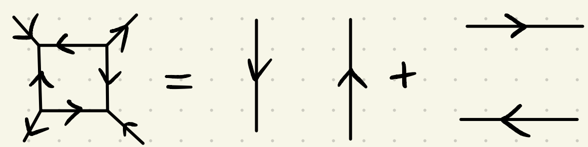
$\mathrm{Hom}_{\mathfrak{sl}_3}(\Bbbk, V^{\otimes 3n})$ has a basis of reduced webs

e.g. $n=2$



reduced: cannot apply local relations

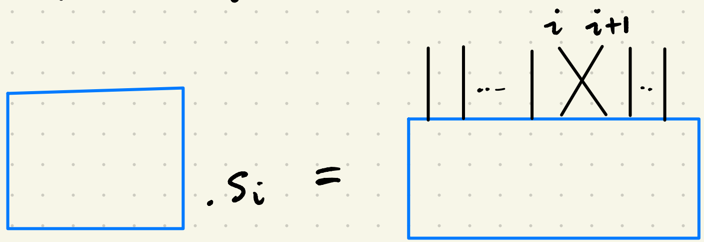




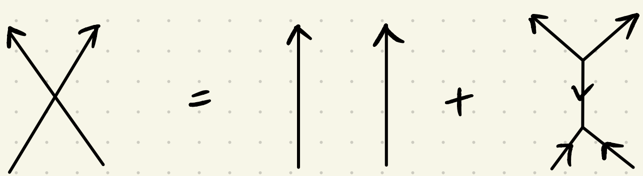
Recall

$\mathrm{Hom}_{\mathfrak{sl}_3}(\Bbbk, V^{\otimes 3n})$: mod-S_{3n}

diagrammatically, this is



subject to the skein relation



3. Transitioning Between the two

Th (Tymoczko)

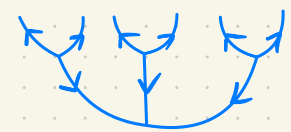
$\exists$ bijection of ordered sets

standard Young tableaux $\overset{\phi}{\longleftrightarrow}$ reduced $\mathfrak{sl}_3$-web

e.g

1	2
3	4
5	6

$\longmapsto$



Th (Russell-Tymoczko)

$\exists$ uni-Δ base change

polytabloid basis $\rightsquigarrow$ reduced web basis

$$V_T = \phi(T) + \sum_{S \leq T} a_{S,T}\, \phi(S)$$

top summand

lower terms

Conj (Russell–Tymoczko) $a_{S,T} \geqslant 0$

for sl_2

Th (Roades) $a_{S,T} \geqslant 0$

Th (Im-Z.) $a_{S,T} >$

4. M-diagram basis

Th (Tymoczko)

$\exists$ bijection

Standard Young Tableaux $\longleftrightarrow$ M-diagrams

" Rank 3 Temperley-Lieb "

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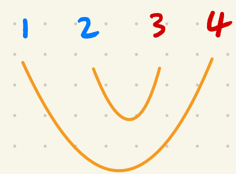
3	4
5	6



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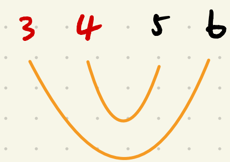

$$V_T = \underbrace{\phi(T)}_{\text{top summand}} + \underbrace{\sum_{S \leq T} a_{S,T}\, \phi(S)}_{\text{lower terms}}$$

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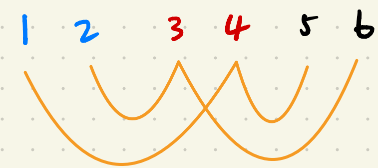

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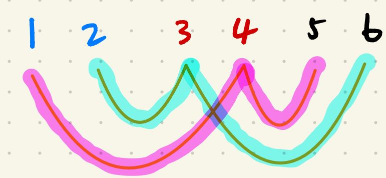
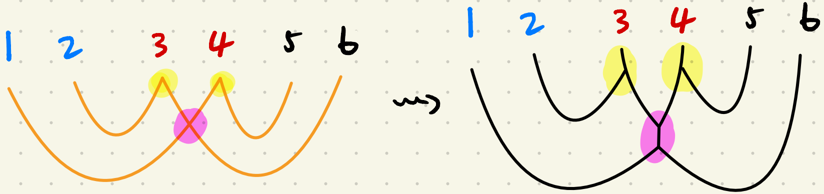


two M's

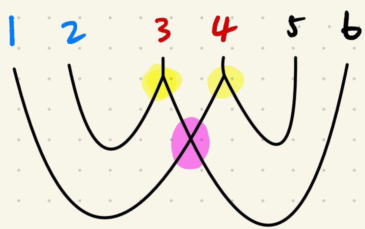


Def M-diagram = crossings preserved

e.g.

1	2
3	4
5	6

$\overset{\psi}{\longmapsto}$



Th (Z.)

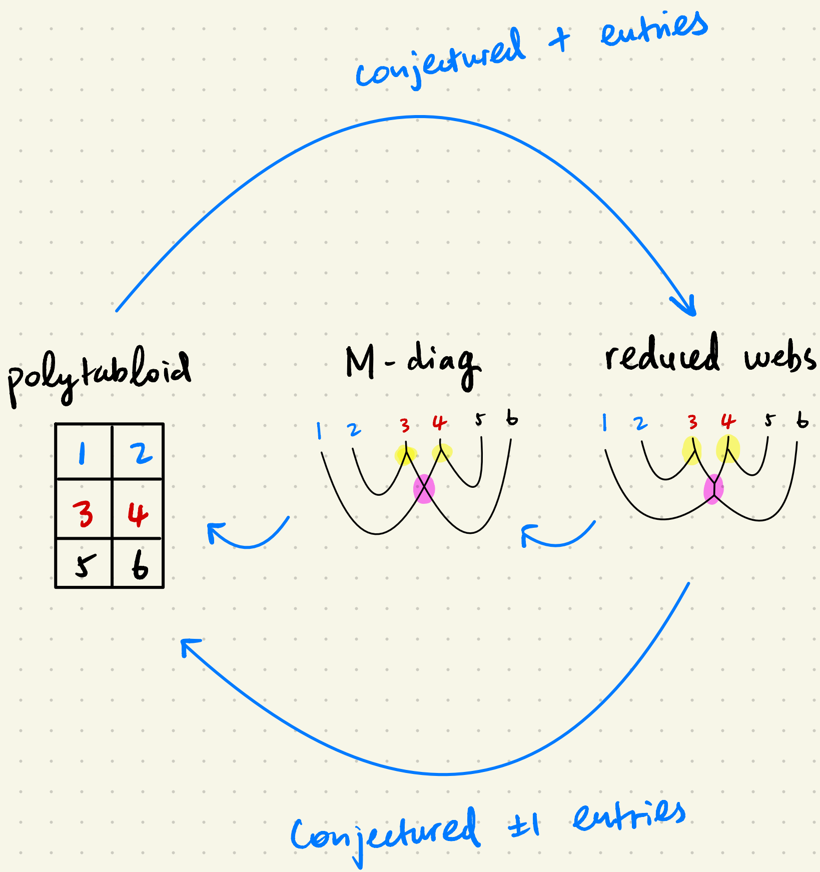
M-diagrams form a basis for D^λ with $\lambda = (n, n, n)$

and $\exists$ uni-Δ base change to polytabloid basis

i.e. $v_T = \psi(T) + \text{lower terms}$

& also uni-Δ to reduced web basis

Motivation



$$\mathrm{Hom}_{sl_3}\left((V^{\pm})^{\otimes p}, (V^{\pm})^{\otimes q} \right)$$

has

cellular basis (Elias)

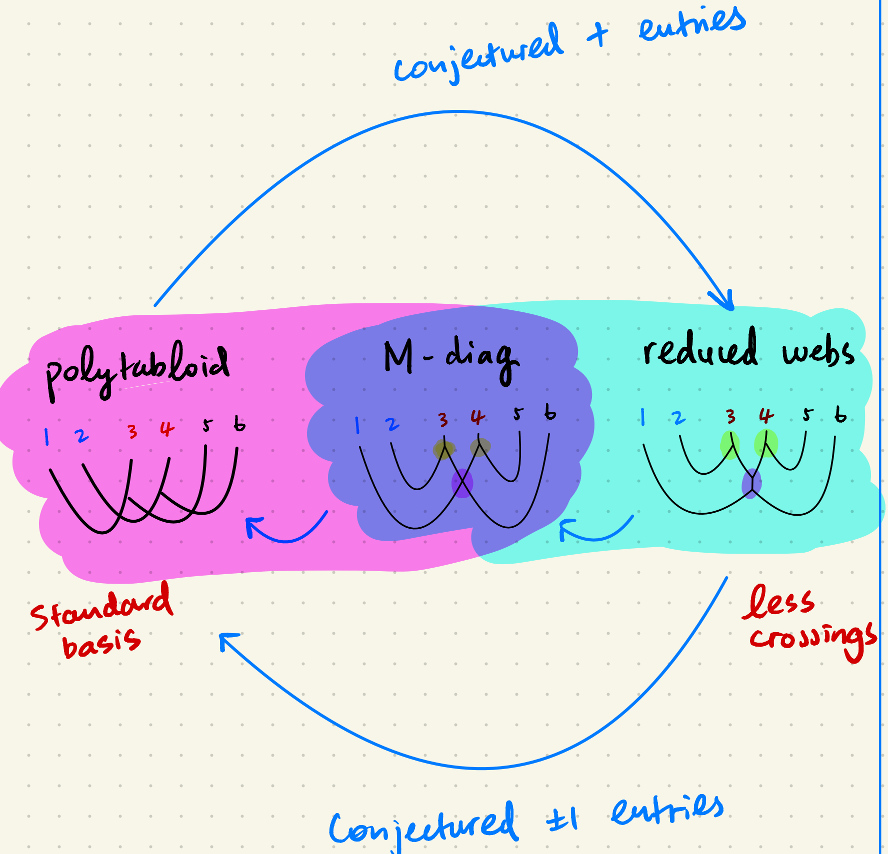
dual-canonical basis (Tubbenhauer)

<u>Q</u> Suitable basis for

Khovanov foam alg

Kazhdan-Lusztig basis in-progress

Motivation



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