Transitioning Between The Tableaux Bases and The Web Bases for The Specht Modules

Mee Seong Im , Jieru Zhu

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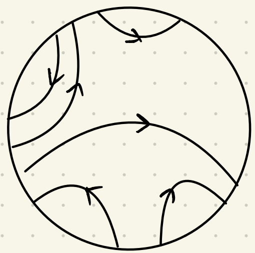
Classic Invariant Theory

sl_2

(Rumer, Teller, Weyl)



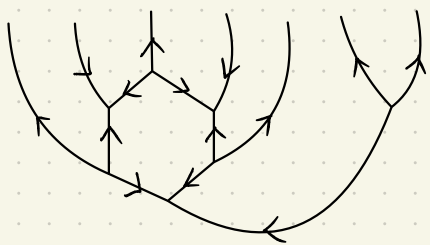
$\in (V^{\pm})^{\otimes 12}$ sl_2-invariant

V^+: natural sl_2-mod

$V^- := (V^+)^*$

sl_3

(Kuperberg '96)

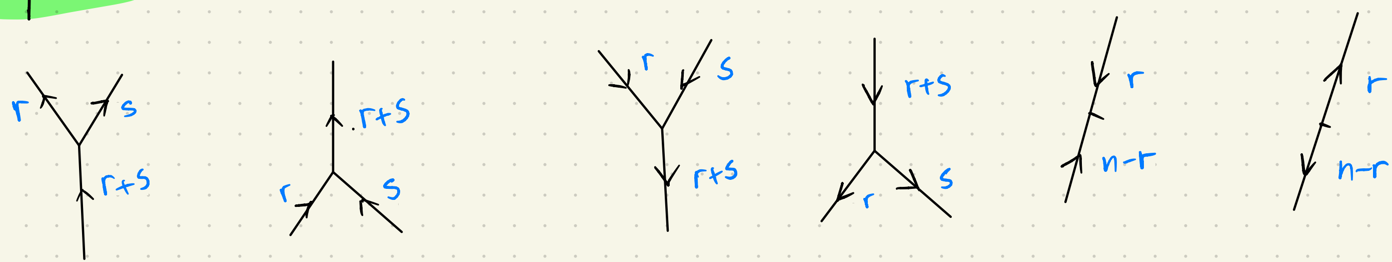


$\in (V^{\pm})^{\otimes 7}$ sl_3-invariants

SL_n- webs (Cautis - Kamnitzer - Morrison)

<u>Definition</u> The category $\mathcal{W}eb_n$ is a $\mathbb{C}(q)$- linear category with

- **object**: sequences of integers
- **morphisms**: vertical & horizontal stacking of diagrams:



$\in \mathrm{Hom}(r+s, (r,s))$

$\in \mathrm{Hom}(-(r+s), (-r,-s))$

$\#$ edges coming in $= \#$ edges coming out

under relations ....

"erase" all edges $\big|\, r$ for $r=0$ or $r=n$

"kill" a picture with $\big|\, r$ for $r>n$

Theorem (Cautis–Kamnitzer–Morrison 2014)

$\boxed{\begin{array}{l} \exists\ \mathbb{C}(q)\text{-linear monoidal functor:} \\ \qquad \mathbb{W}eb_n \longrightarrow U_q(\mathfrak{sl}_n)\text{-reps} \\ \text{(objects)} \qquad r \longmapsto \Lambda_q^r V^+ \\ \qquad\qquad\quad -r \longmapsto \Lambda_q^r V^- \end{array}}$

"MONOIDAL"

e.g.

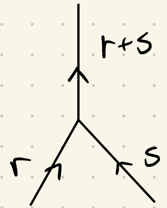
$$(r_1, r_2) \longmapsto \Lambda_q V^+ \otimes \Lambda_q V^+$$

$\boxed{\underline{\text{FULL}} \text{ on a subcategory gen. by } \Lambda_q^s V^\pm}$

i.e. $\mathrm{Hom}_{U_q(\mathfrak{sl}_n)}\left(\Lambda_q^{r_1} V^\pm \otimes \cdots \otimes \Lambda_q^{r_a} V^\pm,\ \Lambda_q^{s_1} V^\pm \otimes \cdots \otimes \Lambda_q^{s_b} V^\pm\right)$

spanned by Webs!

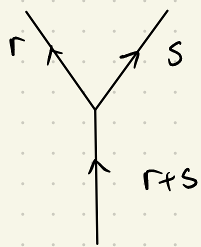
AND **morphisms**



$\longmapsto$ $\left(\textbf{merge} : \quad \Lambda^r V \otimes \Lambda^s V \to \Lambda^{r+s} V \right.$

$$w_1 \wedge w_2 \wedge \cdots \wedge w_r \otimes w_1' \wedge \cdots \wedge w_s'$$

$$\left. \mapsto w_1 \wedge w_2 \wedge \cdots \wedge w_r \wedge w_1' \wedge \cdots \wedge w_s' \right)$$



$\longmapsto$ $\left(\textbf{Split} : \quad \Lambda^{r+s} V \to \Lambda^r V \otimes \Lambda^s V \right.$

$$w_1 \wedge w_2 \wedge \cdots \wedge w_{r+s} \mapsto$$

$$\left. \sum_{\substack{S,T : \text{ partition} \\ \text{of } \{1, \cdots, r+s\} \\ |S| = r \\ |T| = s}} w_{i_1} \wedge w_{i_2} \wedge \cdots \wedge w_{i_r} \otimes w_{j_1} \wedge w_{j_2} \wedge \cdots \wedge w_{j_s} \right)$$

Here $S = \{ i_1 \leq i_2 \leq \cdots \leq i_r \}$

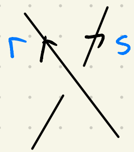
$T = \{ j_1 \leq j_2 \leq \cdots \leq j_s \}$

(Proof)

Check all relations hold!

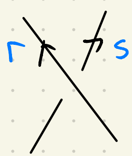
BRAIDING

$\hat{R} \in U_q(sl_n) \otimes U_q(sl_n)$ universal R-matrix i.e. Casimir Tensor

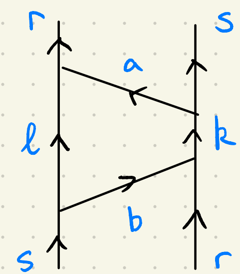


$\in \mathrm{Hom}(\Lambda_q^s V^+ \otimes \Lambda_q^r V^+ , \Lambda_q^r V^+ \otimes \Lambda_q^s V^+)$ by action of $\hat{R}$

(Cautis-Kamnitzer-Morrison)



$= \sum_{l,k,a,b} *$

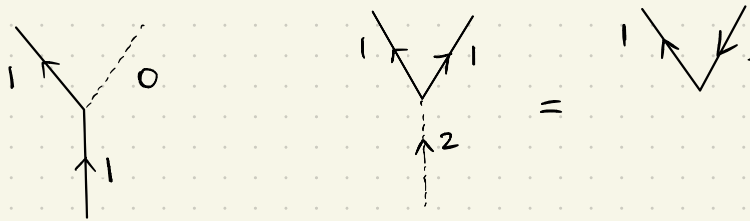


Note: $q \rightsquigarrow 1$ $\quad \hat{R} \rightsquigarrow$ swap map

SPECIAL CASES

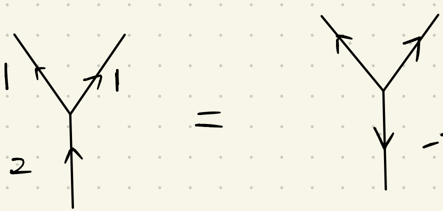
n=2: all strands have thickness 1

e.g.

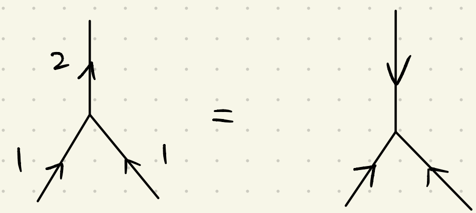


$-1 = 1 \mod 2$

n=3: Kuperberg's spiders



$-2 = 1 \mod 3$



WAIT .... HOW DOES THIS HAVE ANYTHING TO DO WITH THE SYMMETRIC GROUPS ?

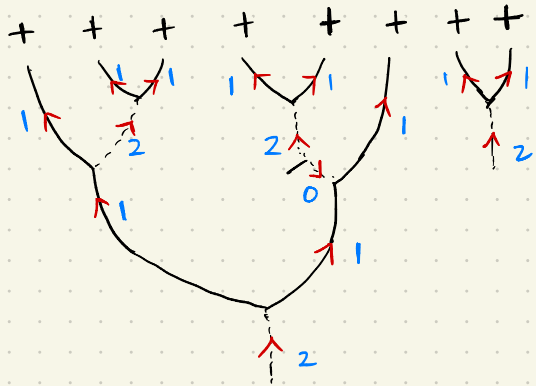
for $n=2$: $\quad V^{\otimes 2d}$: $(\mathfrak{sl}_2, S_{2d})$-bimod $\quad$ $\mathbb{1}$: trivial $\mathfrak{sl}_2$-mod

$\mathrm{Hom}_{\mathfrak{sl}_2}(\mathbb{1}, V^{\otimes 2d})$: S_{2d}-mod

$\llcorner$ space of invariants

$x \in V^{\otimes 2d}$ invariant. $\quad S_{2d} \curvearrowright x$ by permuting tensor factors

e.g.



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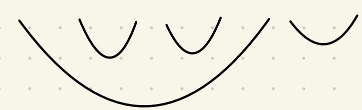
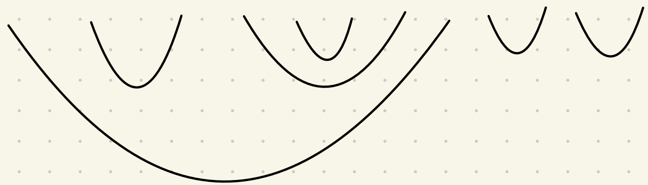


COMBINATORIAL CONSTRUCTION

$\lambda = (d,d)$, $n=2$, S_{2d}.

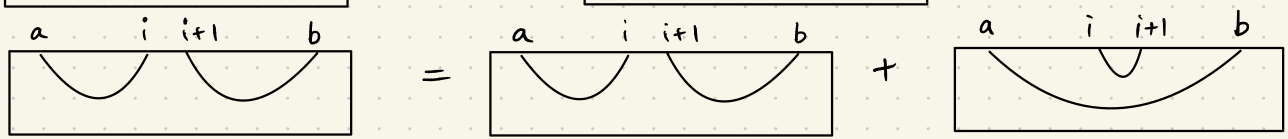
D^λ : $\mathbb{C}$- v.s. with basis: all cup diagrams

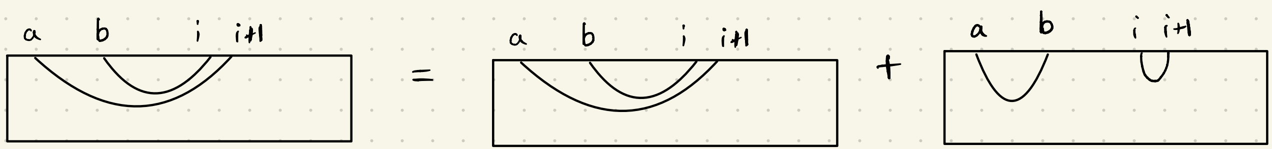


(Non-crossing matchings) with $2d$ boundary pts.

S_{2d} - action :

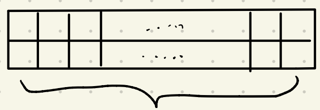
s_i.


s_i.


s_i.


COMBINATORIAL CONSTRUCTION Ver. 2

Young diagram of shape $\lambda = (d, d)$



d boxes

Young Tableau of shape λ with entries $1, 2, \cdots, 2d$

e.g. $d = 5$

6	1	7	8	5
2	10	4	3	9

STANDARD IF: e.g.

1	2	4	7	8
3	5	6	9	10

entries increase → & ↓

column word: 6 2 1 10 7 4 8 3 5

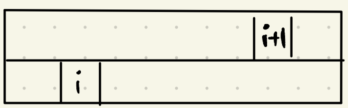
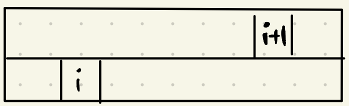
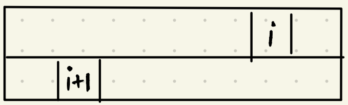
descent of a word: $\#\{(i,j) \mid a_i > a_j,\ 1 \le i < j \le 2d\}$

$\overline{D}^\lambda$: $\mathbb{C}$-v.s spanned by $\{ u_T \mid T$: all Young Tab. $\}$ / linear relations

$S_{2d} \circlearrowright \overline{D}^\lambda$ by permuting entries.

Theorem $\overline{D}^\lambda$ has a basis $\{ v_T = \overline{u}_T \mid T$: standard Young Tab $\}$

Remark $\overline{D}^\lambda$ is irreducible :

$s_i \cdot v$  $= v$  $+ v$ 

column words .... i ---- $i+1$ .... $i+1$ ------ i ----

descent +1

The Tableaux Graph $(n=2,\ d=3)$

(Russell-Tymoczko 2020)

$$w_0 = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline\end{array} \qquad w_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline\end{array} \qquad w_2 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline\end{array}$$

$$w_3 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline\end{array} \qquad w_4 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline\end{array}$$

$$\begin{tikzcd}
& w_0 \arrow[dl, "S_2"'] \arrow[dr, "S_4"] & \\
w_1 \arrow[dr, "S_4"'] & & w_2 \arrow[dl, "S_2"] \\
& w_3 \arrow[d, "S_3"] & \\
& w_4 &
\end{tikzcd}$$

Def

$$T_0 = \begin{array}{|c|c|c|c|}\hline 1 & 3 & \cdots & 2d-1 \\ \hline 2 & 4 & \cdots & 2d \\ \hline\end{array}$$

$R \leq S$

$\Longleftrightarrow$ $\exists$ directed path from R to S

Refinement of

"descent of column word"

Module Isomorphism

Th $\bar{D}^\lambda \cong D^\lambda$ (Specht module corresponding to cycle type λ)

and up to a scalar,

$v_{T_0} =$  $=: w_0$

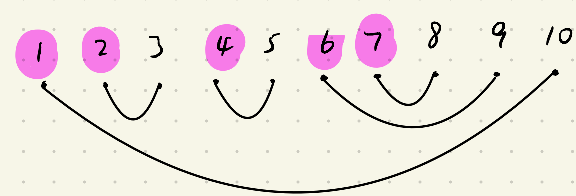
In fact, $\dim \bar{D}^\lambda = \dim D^\lambda = d^{\text{th}}$ Catalan number

$$\phi : \{\text{standard Young Tableaux}\} \xrightarrow{\sim} \{\text{cup diagrams}\}$$

e.g.

1	2	4	6	7
3	5	8	9	10

$\mapsto$



"Catalan identification"

Unitriangularity Result

Transfer "$\leq$" on {standard Young Tab} to {cup diag}

Th Russell – Tymoczko (2020)

$$v_T = \phi(T) + (\text{terms} \leq \phi(T))$$

Conjecture (Russell – Tymoczko) POSITIVITY

- Lower terms have positive coeff
- all lower terms occur

Nonnegativity Result

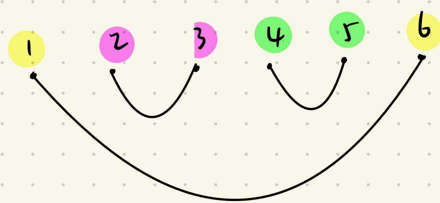
Th (Rhoades 2020)

- all lower terms have NONNEGATIVE coefficients

Idea of Proof:

$M = [\,\vec{v}_1\ \vec{v}_2 \cdots \quad \vec{v}_{2d}\,]$ $2 \times 2d$ matrix $\circlearrowleft$ S_{2d} permutes columns

$\{\text{cup diag}\} \longrightarrow \{\text{products of minors of } M\}$

e.g. 

$\mapsto \det[\vec{v}_1\ \vec{v}_6]\ \det[\vec{v}_2\ \vec{v}_3]\ \det[\vec{v}_4\ \vec{v}_5]$

Relations are equivalent

Tableaux

(Garnir Relations)

$$v\ \begin{array}{|c|c|c|c|}\hline \quad & a & b & \quad \\ \hline \quad & c & d & \quad \\ \hline\end{array} = v\ \begin{array}{|c|c|c|c|}\hline \quad & a & c & \quad \\ \hline \quad & b & d & \quad \\ \hline\end{array} + v\ \begin{array}{|c|c|c|c|}\hline \quad & a & b & \quad \\ \hline \quad & d & c & \quad \\ \hline\end{array}$$

$$= \left(v\ \begin{array}{|c|c|c|c|}\hline \quad & b & a & \quad \\ \hline \quad & c & d & \quad \\ \hline\end{array} \right)$$

Plücker relations

for minors

$$\begin{bmatrix} \vec{v}_a & \vec{v}_b & \vec{v}_c & \vec{v}_d \end{bmatrix} : \quad 2 \times 2d \text{ matrices}$$

$$\det\left[\vec{v}_a \vec{v}_c\right] \cdot \det\left[\vec{v}_b \vec{v}_d\right] = \det\left[\vec{v}_a \vec{v}_b\right] \det\left[\vec{v}_c \vec{v}_d\right] + \det\left[\vec{v}_a \vec{v}_d\right] \det\left[\vec{v}_b \vec{v}_c\right]$$

Relations are equivalent

Tableaux

(Garnir Relations)

$$v \begin{array}{|c|c|c|c|}\hline & a & b & \\ \hline & c & d & \\ \hline \end{array} = v \begin{array}{|c|c|c|c|}\hline & a & c & \\ \hline & b & d & \\ \hline \end{array} + v \begin{array}{|c|c|c|c|}\hline & a & b & \\ \hline & d & c & \\ \hline \end{array}$$

$$= \left(v \begin{array}{|c|c|c|c|}\hline & b & a & \\ \hline & c & d & \\ \hline \end{array} \right)$$

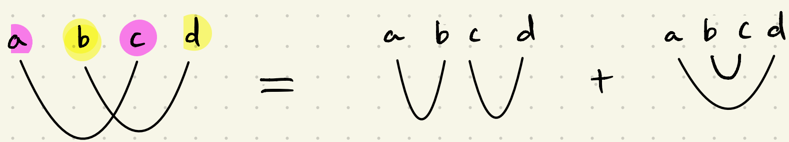
Plücker relations

for minors

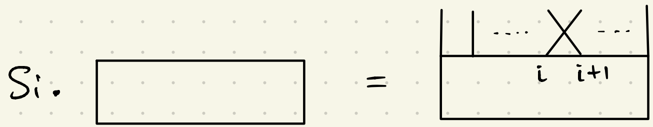
$$\left[\vec{v}_a \quad \vec{v}_b \quad \vec{v}_c \quad \vec{v}_d \right] : \quad 2 \times 2d \text{ matrices}$$

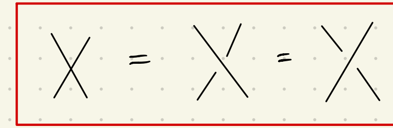
$$\det[\vec{v}_a \vec{v}_c] \cdot \det[\vec{v}_b \vec{v}_d] = \det[\vec{v}_a \vec{v}_b] \det[\vec{v}_c \vec{v}_d] + \det[\vec{v}_a \vec{v}_d] \det[\vec{v}_b \vec{v}_c]$$

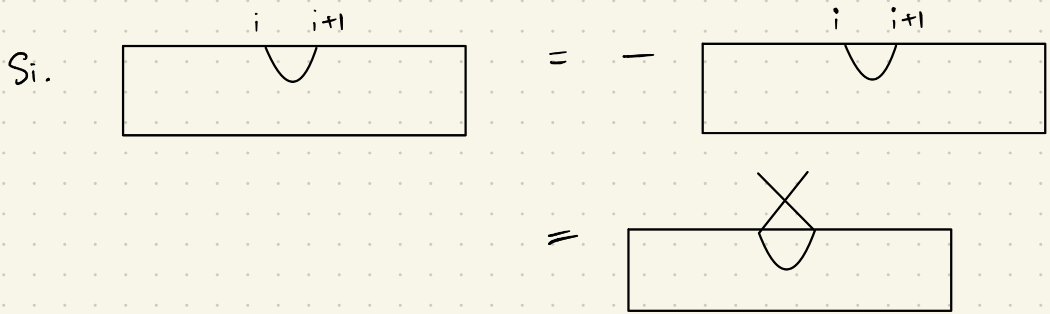
diagrams with crossings

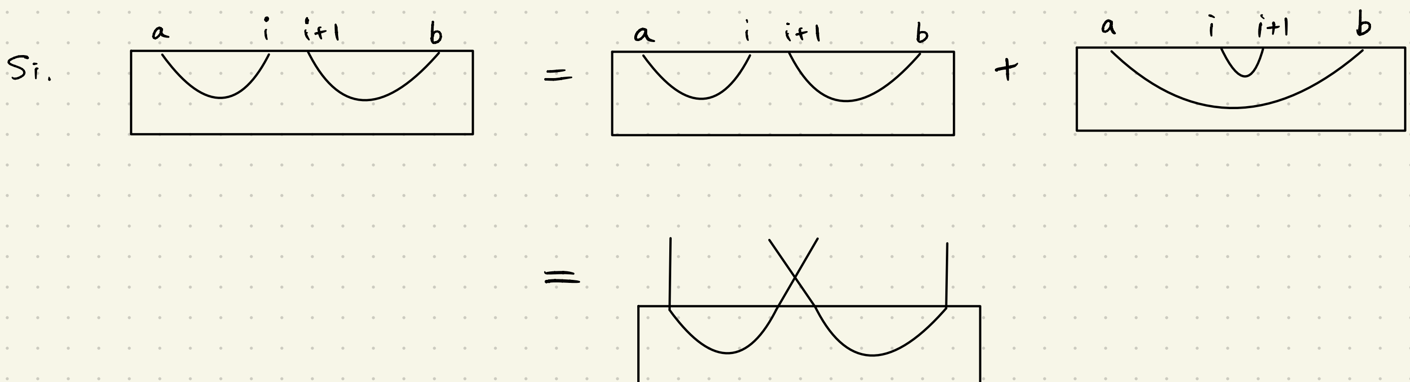


Nonnegativity Result, reformulated

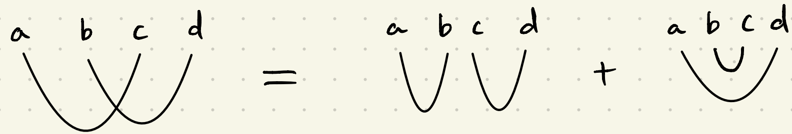


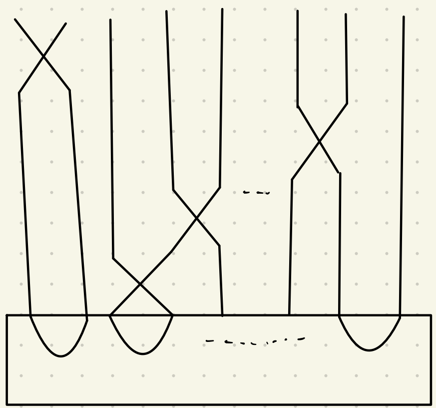






Braid on top of W_0

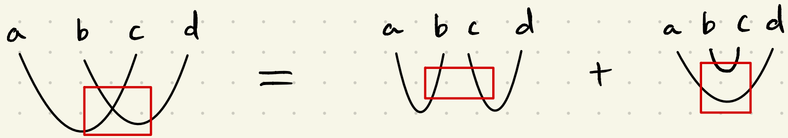


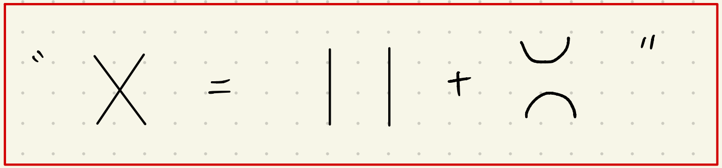


Straigten (Up to Isotopy)

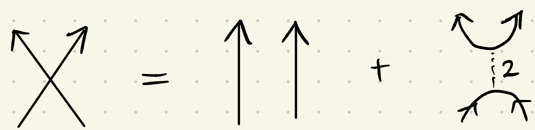
Then expands positively !

The Skein Relations



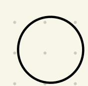


Specialization of skein relation for webs when $q \rightsquigarrow 1$



Braiding in $\mathrm{Rep}\,\mathfrak{sl}_2$

in $\mathbb{W}eb_2$

Warning! This relation is LOCAL, " $= -2$"

A web for A Tableau

Prop. (Im-Z. 2021)

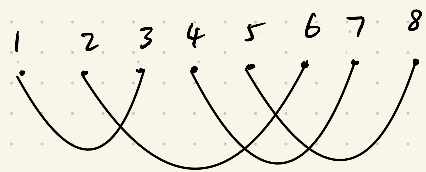
In $\bar{D}^\lambda \simeq D^\lambda$

$v_T \mapsto$ The diagram whose arcs join two entries in the same column

e.g. v

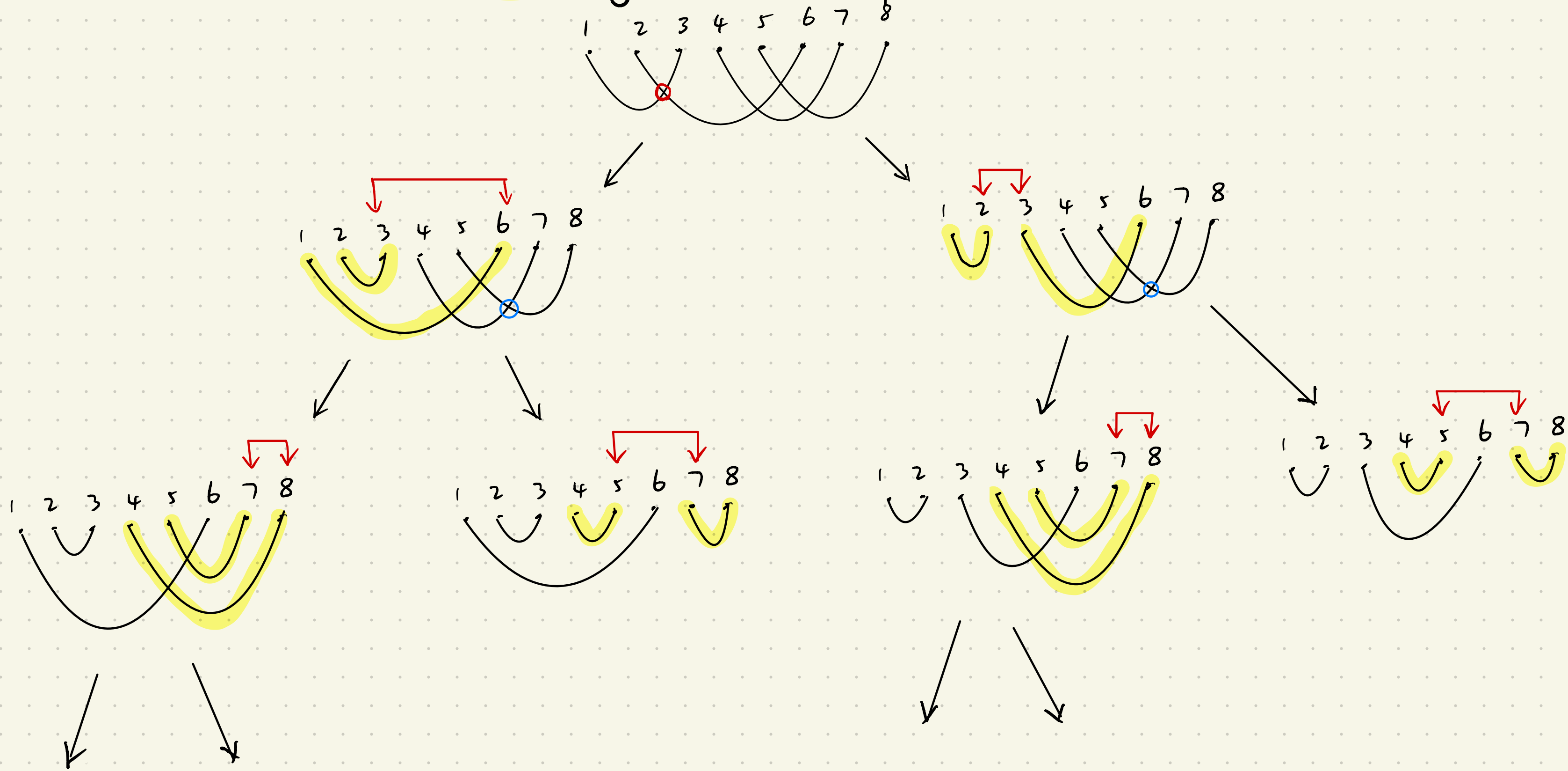
1	2	4	5
3	6	7	8

$\mapsto$



"column identification"

Crossing Resolving Graph



...... until no crossings remain!

Existence of Path

Observation 1) "Repetitions" coeff of a cup diag D = # path from the top to D

2) D occurs in the expansion $\Leftrightarrow$ $\exists$ a path to D

Remark. This graph is NOT uniquely defined.

Propostion (Im-Z.) (Reformation)

if S, T standard Young Tab. $S \leq T$

$$T \overset{(\text{column ident.})}{\updownarrow} w_1 \qquad S \overset{(\text{Catalan ident.})}{\updownarrow} w_2$$

Then $\exists$ path $w_1 \to w_2$ in <u>one</u> crossing-resolving graph

Proof (by example)

s_5

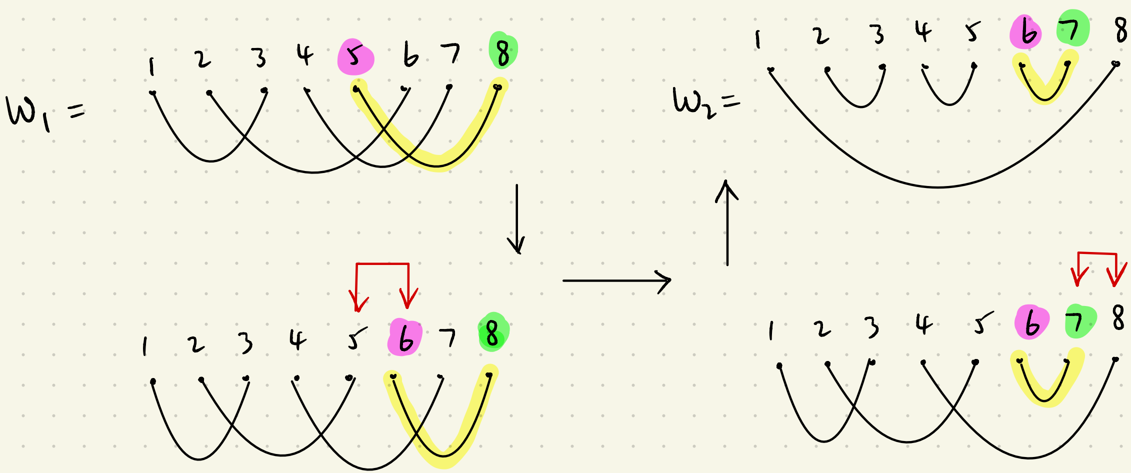
$T =$

1	2	4	5
3	6	7	8

$S =$

1	2	4	6
3	5	7	8

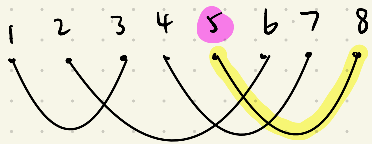
$5 < 6$



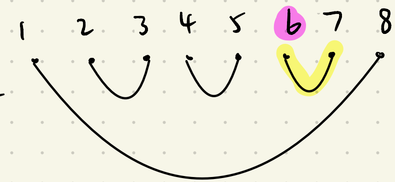
Induction

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 & 6 & 7 & 8 \\ \hline \end{array} \xleftarrow{\;s_5\;} S = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 6 \\ \hline 3 & 5 & 7 & 8 \\ \hline \end{array}$$

$w_1 =$

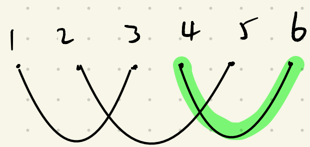


$w_2 =$

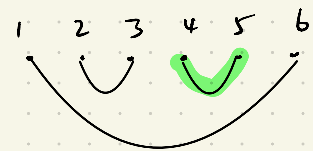


$$T' = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array} \qquad S' = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array}$$

$w_1' =$



$w_2' =$



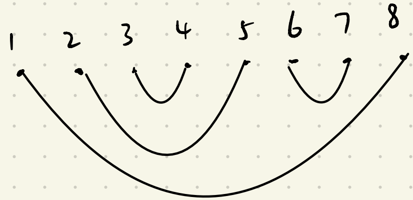
The Inverse Map

Prop (Im-Z.)

If W: cup diag, then $W = V_T$, where

T: the (nonstandard) Young Tab given by the

"column identification"

e.g. $W =$ 

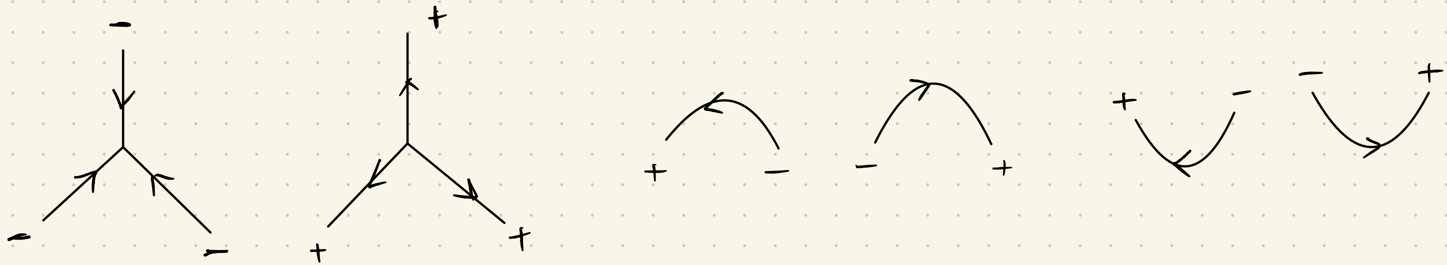
$T =$

1	2	3	6
8	5	4	7

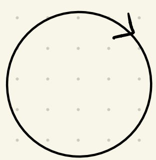
Conjecture W expands into V_T's with coeff $\pm 1, 0$

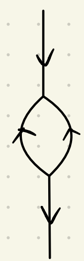
SL_3- webs

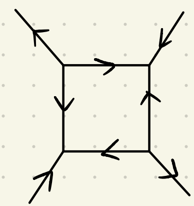
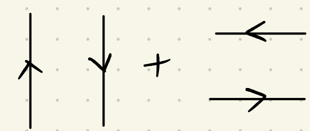
(Kuperberg) Trivalent graphs gen. by



under relations

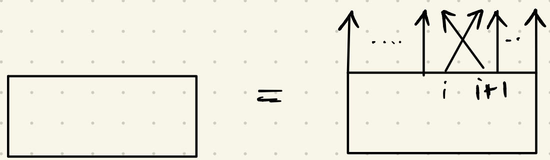
 $= 3$

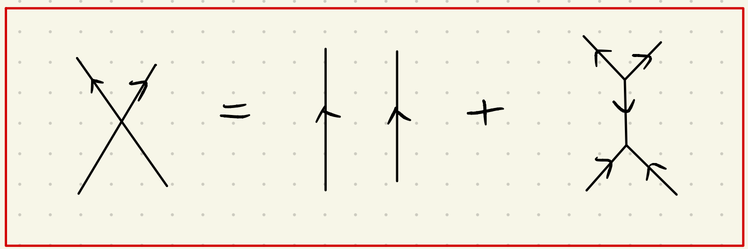
 $= -2$

 $=$ 

$\lambda = (d,d,d)$

$D^\lambda = \mathrm{Hom}_{sl_3}(\mathbb{1}, (V^+)^{\otimes 3d})$ = space spanned by all sl_3-webs with $3d$ $+$'s on top

$S_i \cdot$ 

Via 

applied LOCALLY

Th (Petersen-Pylyavskyy-Rhoades 2009) (Im-Z.)

$D^\lambda \cong \bar{D}^\lambda$ Specht module for λ

Moreover, $\quad V_{T_0} = W_0$

$T_0 =$

1	4	...
2	5	..
3	6	...

$W_0 =$ 

Sketch of Proof

① Jucys-Murphy elts $\curvearrowright$ Gel'fand - Zetlin basis element by scalars

② JM $\curvearrowright$ V_{T_0}, W_0 by scalars = contents of boxes in λ

Specht Module for $\lambda \subseteq \mathrm{Hom}_{sl_3}(\mathbb{1}, V^{\otimes 2d})$

③ $\dim($Specht Module for $\lambda) = \dim \mathrm{Hom}_{sl_3}(\mathbb{1}, V^{\otimes 2d})$ Bratteli graph.

Existing Literature / Future directions

Kuperberg-Khovanov (98) $\{$ standard YT of shape $\lambda \}$ $\longleftrightarrow$ $\{$ webs $\}$

"growth algorithm"

Tymoczko "m diagrams"

Russell-Tymoczko (2020) "$\leq$" on webs

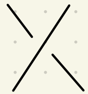
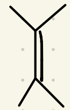
& unitriangularity

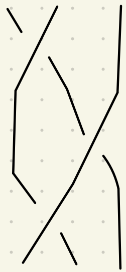
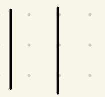
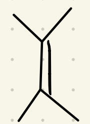
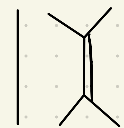
Missing Positivity ?

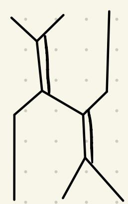
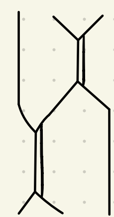
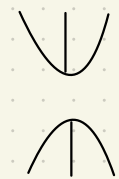
Quantum case? (Monomials ?)

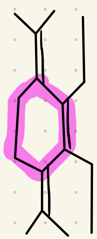
Higher "n" in $\mathfrak{sl}_n$?

Some Ideas for sl_n

$H_i =$  $=$  $+$ $= \underline{H}_i + 1$

$H_i H_{i+1} H_i =$  $=$  $+$  $+$  $+$

 $+$  $+$ 

"  " Bott-Samuelson basis

$= ? + ?\underline{H}_i + ?\underline{H}_{i+1} + ?\underline{H}_{s_i s_{i+1}} + ?\underline{H}_{s_{i+1} s_i} + ?\underline{H}_{s_i s_{i+1} s_i}$

? inverse KL poly's

CLOSE off The Braid with forks (n=3)