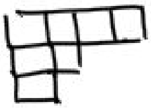
Transitioning Between Tableaux & Web Bases of The Specht Module (jt w/ Mee Seong Im)

S_d Sym. gp

$\lambda \vdash d$ partion

$\lambda = (4,2,1)$ $\qquad \lambda =$ 

$$\begin{array}{|c|c|c|c|}\hline 1 & 7 & 3 & 6 \\ \hline 4 & 2 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \qquad\qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 7 \\ \hline 3 & 6 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$$

nonstandard $\qquad$ Standard Young tableau of shape λ

$S^\lambda := \langle v_T \mid T : \text{Young } T \rangle / \sim$ "Garnir Relations"

$\circlearrowright$ S_d permuting entries

Th S^λ has a basis $\{ v_T \mid T : \text{Stand. Y.T} \}$

$\lambda = (n, n)$

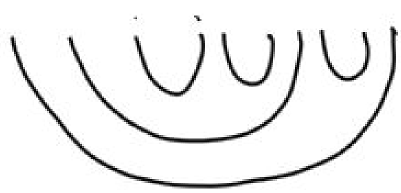
$sl_2(\mathbb{C}) \curvearrowright V = \mathbb{C}^2$ $\qquad V^{\otimes 2n}$ (sl_2, S_{2n})-bimod

$\curvearrowright V^{\otimes 2n} \curvearrowleft S_{2n}$ $\quad \mathbb{C}$

$\mathrm{Hom}_{sl_2}(\mathbb{C}, V^{\otimes 2n})$ $\qquad S_{2n}$-mod

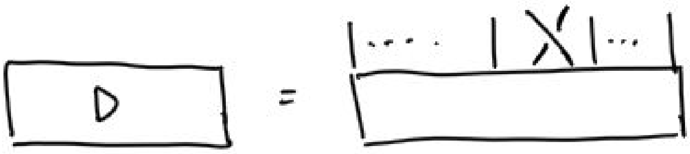
$f \longleftrightarrow f(1) \in V^{\otimes 2n}$

Th $\mathrm{Hom}_{sl_2}(\mathbb{C}, V^{\otimes 2n}) = \langle \text{cup diagrams} \rangle$

2n dots / n arcs  $\in V^{\otimes 2n}$

coev: $\mathbb{C} \to V \otimes V^*$ is $V \otimes V$

$1 \mapsto v_1 \otimes v_1^* + v_2 \otimes v_2^*$

s_i .  ← web

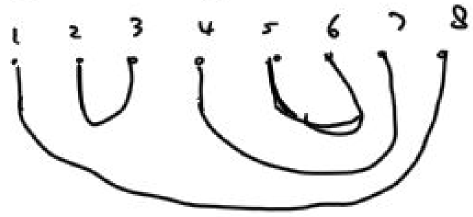
↑ cup diagrams

Fact $\underbrace{S^{(n,n)}} \simeq \underbrace{\mathrm{Hom}_{\mathfrak{sl}_2}(\mathbb{C}, V^{\otimes 2n})}$

ϕ: $\{\text{S.Y.T}\} \longleftrightarrow \{\text{cup diag.}\}$

$T \longmapsto$ diag. whose endpoints are in the first row of T

$\begin{matrix} 1 & 2 & 4 & 5 \\ 3 & 6 & 7 & 8 \end{matrix}$



T. vs v_T

$w_T = \cancel{\phi(w)}\ \phi(T)$

~~$\{v_T\}$~~

$\langle v_T \rangle \xrightarrow[\sim]{\varphi} \langle w_T \rangle$

$T_0 = \begin{matrix} 1 & 3 & 5 & \cdots & 2n-1 \\ 2 & 4 & 6 & \cdots & 2n \end{matrix}$ $\phi(T_0) = \cup \cup \cdots \cup$.

$\varphi(v_{T_0}) = w_{T_0}$

T: S.Y.T. $\underbrace{T} = \underbrace{w} T_0$ $w \in S_d$

on S_d "$\leq$" weak Bruhat order

"$\leq$" S.Y.T

(Russell-Tymoczko '19) $\varphi(v_T) = w_T +$ lower terms

$\varphi(v_T) = \sum_S \underbrace{a_{S,T}} w_S$

$(a_{S,T})$ is upper Δ, $a_{S,S} = 1$

(Rhoades '19) $a_{S,T} \geq 0$

(Im-Z.) $a_{S,T} \geq 0$ $\varphi(V_T) = w_T + \text{lower}$

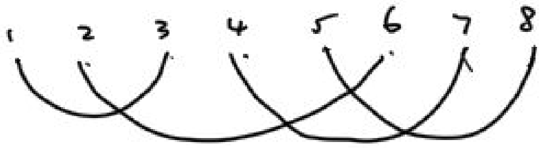
Proof by Rhoades:

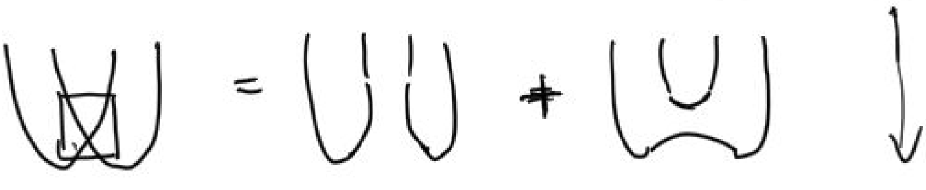
$\varphi(w.\underbrace{V_{T_0}}) = w\,\underbrace{\varphi(V_{T_0})} = w. \cup\cup \cdots \cup$

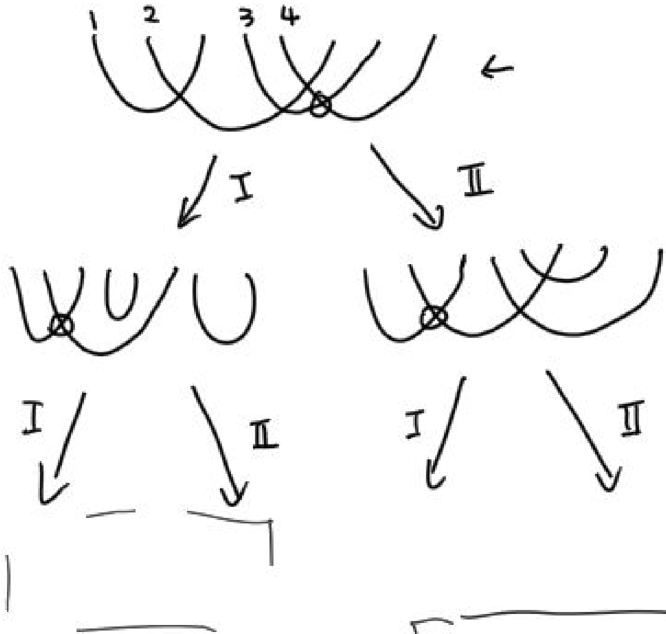
$\underline{a_{S,S} = 1}$

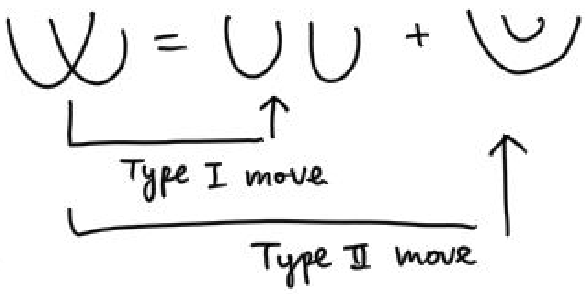
<u>Lemma</u> $\varphi(V_T) =$ web s.t. each column of T is joined by an arc

$\varphi\left(V_{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 & 6 & 7 & 8 \\ \hline\end{array}}\right) =$



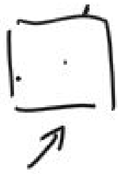


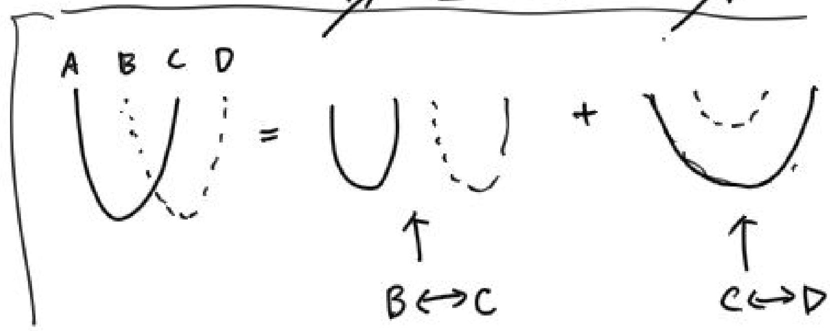




$a_{S,T}$

$= \sum\limits_{\text{colorings}}$ # paths from colored $\underbrace{\rho(V_{T_\bullet})}$ to colored S

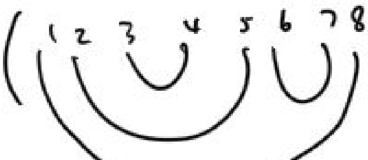




Prop. $\underbrace{a_{S,T}} > 0 \quad \forall\, S < T$

$\exists$ one coloring of S s.t. there is a path

Prop. $\rho^{-1}(w_R) = \underbrace{V_T}$. T is the tableau in which each column is joined by an arc in R

$\rho^{-1}\Big($  $\Big) = V$

$$\underbrace{\begin{matrix} 1 & 2 & 3 & 6 \\ 8 & 5 & 4 & 7 \end{matrix}}_{\downarrow}$$

Conj $(a_{S,T})^{-1}$ has entries $\pm 1, 0$

(Stroppel - Wilbert ~~&~~ '16) B/c in this holds

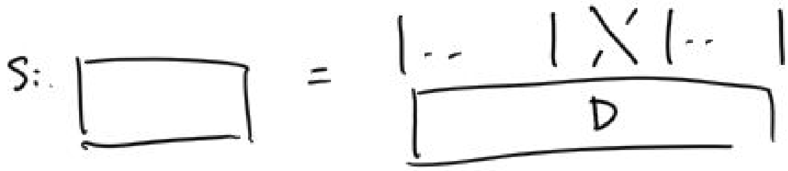
$\lambda = (n,n,n)$

$\mathfrak{sl}_3(\mathbb{C}) \curvearrowright V = \mathbb{C}^3$

$V^{\otimes 3n}$: $(\mathfrak{sl}_3, S_{3n})$ - bimod

(Kuperberg '96) $\mathrm{Hom}_{\mathfrak{sl}_3}(\mathbb{C}, V^{\otimes 3n}) = \langle \text{spiders} \rangle \circlearrowleft S_{3n}$

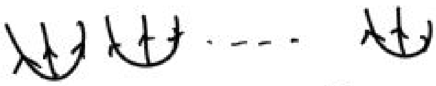
trivalent graphs w/ orient. on edges, $\underbrace{\text{labels}}_{\text{"+"}}$ on bndry pts



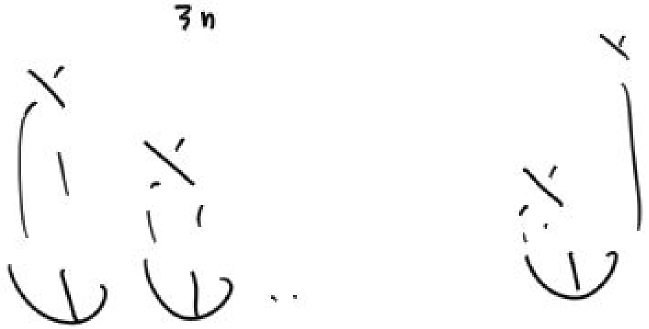
$\leftarrow$ (Petersen - Pylyavskyy - Rhodes '08)

(Cautis - Kamnitzer - Morrison '14)

$\underset{\mathfrak{sl}_3}{\mathrm{Hom}}(\mathbb{C}, V^{\otimes 3n}) \cong \langle \underset{\text{spiders}}{\cancel{\text{webs}}} \rangle \underset{\cong}{\xleftarrow{\varphi}} S^{(n,n,n)}$

Prop. $\varphi\left(V_{\begin{smallmatrix} 1 & 4 & & 3n-2 \\ 2 & 5 & \cdots & 3n-1 \\ 3 & 6 & & 3n \end{smallmatrix}}\right) =$ 

"n claw"

$\varphi(V_T) =$ 

$\varphi(V_T) = \sum_R a_T^R w_R$

fix T either $a_T^R \geq 0 \quad \forall R$.

$a_T^R \leq 0 \quad \forall R$.