Two boundary centralizer algebras for $\mathfrak{q}(n)$

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The Type Q Lie superalgebra

$$\mathfrak{q} = \mathfrak{q}(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} | A, B \in \mathrm{Mat}_{n,n}(\mathbb{C}) \right\}$$

$$\mathfrak{q}_{\overline{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} | A \in \mathrm{Mat}_{n,n}(\mathbb{C}) \right\},$$

$$\mathfrak{q}_{\overline{1}} = \left\{ \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} | B \in \mathrm{Mat}_{n,n}(\mathbb{C}) \right\}.$$

with

$$[x, y] = xy - (-1)^{\overline{x} \cdot \overline{y}} yx$$

$V = \mathbb{C}^{2n}$: $\mathfrak{q}(n)$-module
action: matrix multiplication on the left.
M, N: $\mathfrak{q}(n)$-module

$$\begin{aligned}\Delta : U(\mathfrak{q}(n)) &\to U(\mathfrak{q}(n)) \otimes U(\mathfrak{q}(n))\\ x &\mapsto 1 \otimes x + x \otimes 1\end{aligned}$$

$$\mathfrak{q}(n) \curvearrowright M \otimes N \otimes V^{\otimes d} \curvearrowleft ??$$

A superalgebra $S = S_{\overline{0}} \oplus S_{\overline{1}}$,

$$S_{\overline{i}} \cdot S_{\overline{j}} \subset S_{\overline{i+j}}, \qquad \overline{i}, \overline{j} \in \mathbb{Z}_2$$

The degenerate two boundary affine Hecke-Clifford algebra $\mathcal{B}_d$:
generated by
1) polynomial rings $\mathbb{C}[\tilde{x}_1, \dots, \tilde{x}_d]$, $\mathbb{C}[\tilde{y}_1, \dots, \tilde{y}_d]$, $\mathbb{C}[\tilde{z}_0, \tilde{z}_1, \dots, \tilde{z}_d]$
(polynomial generators: odd)
2) odd $c_1, \dots, c_d$
3) even $s_1, \dots, s_{d-1}$

subject to

1) symmetric group relations, polynomial ring relations

2) $c_i^2 = -1$, $c_i c_j = -c_j c_i$ (Clifford relations).

3) "Hecke" relations such as

$$s_i \tilde{x}_i = \tilde{x}_{i+1} s_{i+1} + (c_i - c_{i+1})$$

4) "Two boundary" type relations such as

$$(\tilde{z}_0 - \tilde{z}_1 - \cdots - \tilde{z}_i + \tilde{x}_i)\tilde{x}_1 = -\tilde{x}_1(\tilde{z}_0 - \tilde{z}_1 - \cdots - \tilde{z}_i + \tilde{x}_i)$$

Theorem (Super Schur's Lemma)

If W, U: simple $\mathfrak{g}$-module, then

$$\mathrm{Hom}_{\mathfrak{q}(n)}(W,U) = \begin{cases} \mathbb{C}\,\mathrm{id} & \text{if } W \simeq U \text{ of Type } M \\ \mathbb{C}\,\mathrm{id} \oplus \mathbb{C}c & \text{if } W \simeq U \text{ of Type } Q \\ 0 & W \not\simeq U \end{cases}$$

where $c \in \mathrm{End}_{\mathfrak{q}(n)}(W)$ is an odd map.

Moreover, $V = \mathbb{C}^{2n}$ is of Type Q, and the map c can be taken as left multiplication by

$$c = \begin{bmatrix} 0 & -1_n \\ 1_n & 0 \end{bmatrix}$$

A Casimir element $\Omega \in \mathfrak{q}(n) \otimes \mathfrak{q}(n)$
Ω acts on $T_1 \otimes T_2$

$$x.(\Omega.(t_1 \otimes t_2)) = (-1)^{\overline{x}\cdot\overline{\Omega}}\Omega.(x.(t_1 \otimes t_2))$$

$x \in \mathfrak{g}$ homogeneous

Lemma (Z.)

$$\Omega^2 = \frac{1}{3}(\Delta(z_2) - z_2 \otimes 1 - 1 \otimes z_2 + 2z_1 \otimes z_1).$$

An action of $\mathcal{B}_d$

Theorem (Z)

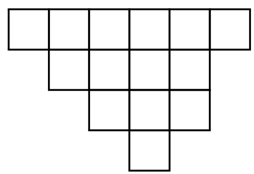
Let M, N be arbitrary $\mathfrak{q}(n)$-modules. There is a superalgebra homomorphism:

$$\begin{aligned} &\mathcal{B}_d \to \mathrm{End}_{\mathfrak{q}(n)}(M \otimes N \otimes V^{\otimes d}) \\ &\tilde{x}_i \mapsto \Omega_{M \otimes V^{i-1}, V} \\ &\tilde{y}_i \mapsto \Omega_{N \otimes V^{i-1}, V} \\ &\tilde{z}_i \mapsto \Omega_{M \otimes N \otimes V^{i-1}, V} \qquad (1 \le i \le d) \\ &\tilde{z}_0 \mapsto \Omega_{M,N} \end{aligned}$$

and s_i acts as signed swap, c_i acts as the map c on the i-th factor.

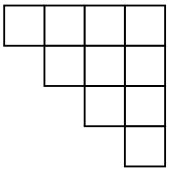
(Sergeev) $V^{\otimes d}$, (Hill-Kujawa-Sussan) $M \otimes V^{\otimes d}$

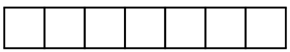
Polynomial representations of $\mathfrak{q}(n)$:
Direct summands of $V^{\otimes e}$ for some $e \in \mathbb{Z}_{\geq 0}$.
This subcategory is semisimple, closed under $\otimes$.
(Berele-Regev, Sergeev) Irreducibles $L(\lambda)$: parametrized by "shifted Young diagram" λ

$\lambda =$ 

$$c(\square) = \text{column of } \square - \text{row of } \square$$

$$\ell(\lambda) = \#\text{ rows in } \lambda$$

Let $\alpha =$  (staircase of height n)

$\beta =$  (single row with p boxes)

Notice: a box added to α has content n
a box added to β has content p or 0

An action of $\mathcal{H}_n^p$

$\mathcal{H}_d^p = \mathcal{B}_d^N / \sim$ under further relations

$$(\tilde{x_1})^2 = n(n+1)$$
$$(\tilde{y_1})^2((\tilde{y_1})^2 - p(p+1)) = 0$$

Theorem

The action of $\mathcal{B}_d$ factors through $\mathcal{H}_n^p$ and induces further a homomorphism of superalgebras

$$\mathcal{H}_n^p \to \mathrm{End}_{\mathfrak{q}(n)}(L(\alpha) \otimes L(\beta) \otimes V^{\otimes d})$$

$$L(\lambda) \otimes L(\mu) \simeq \bigoplus_{\gamma} L(\gamma)^{\oplus c^{\gamma}_{\lambda,\mu}}$$

(Brundan, Stembridge) Combinatorial formula for $c^{\gamma}_{\lambda,\mu}$.
(Bessenrodt) Cases when $c^{\gamma}_{\lambda,\mu}$ is as small as possible.

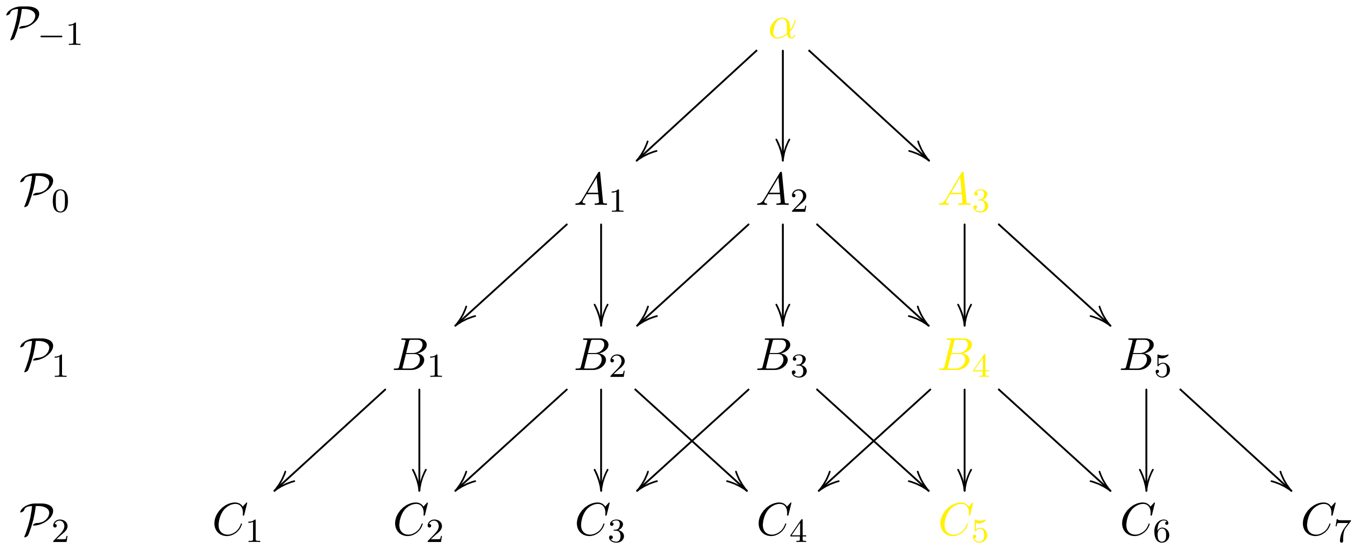
The Bratteli diagram: a directed graph

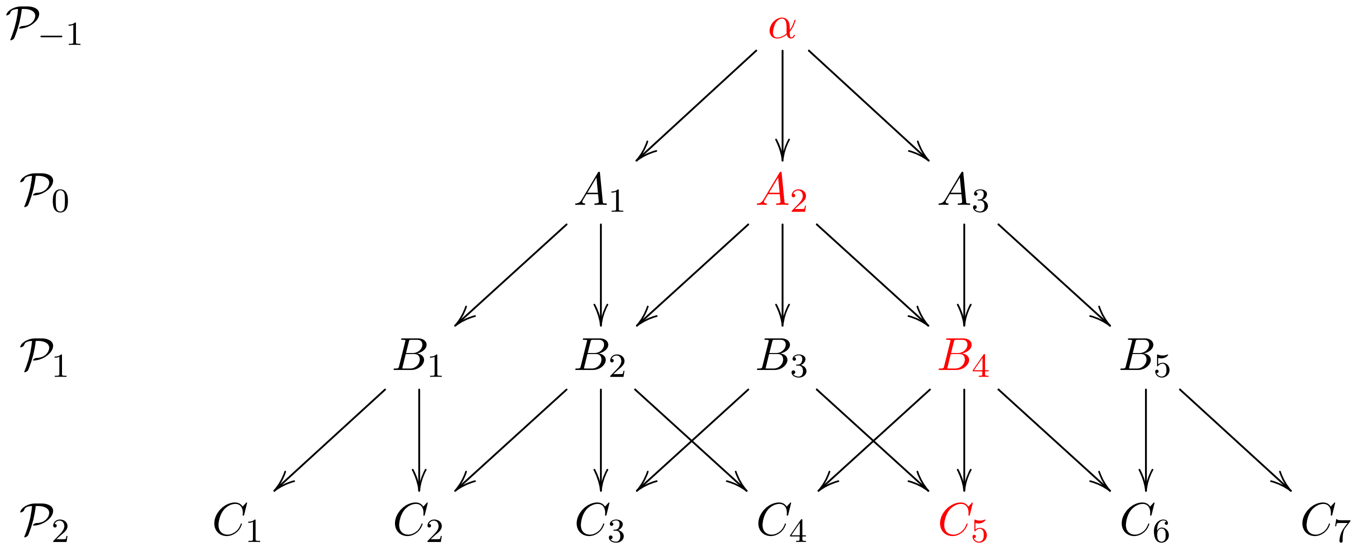
Vertices $\cup_{i=-1}^{\infty}\mathcal{P}_i$

$$\mathcal{P}_{-1} = \{\alpha\}$$

$$\mathcal{P}_0 = \{\gamma \mid L(\gamma) \text{ is a summand of } L(\alpha)\otimes L(\beta)$$
$$\text{(with multiplicity 2)}\}\text{(combinatorial condition)}$$
$$\alpha \to \mu, \forall \mu \in \mathcal{P}_0$$

$$\mathcal{P}_i = \{\gamma \mid L(\gamma) \text{ is a summand of } L(\lambda)\otimes V, \quad \exists \lambda \in \mathcal{P}_{i-1}$$
$$\text{(with multiplicity 2)}\}$$
$$= \{\gamma \mid \gamma = \lambda + \square, \quad \exists \lambda \in \mathcal{P}_{i-1}\}\text{(Pieri Rule)}$$
$$\lambda \to \gamma \text{ if } \gamma = \lambda + \square$$


$\mathcal{P}_{-1}$
α
$\mathcal{P}_0$
A_1 A_2 A_3
$\mathcal{P}_1$
B_1 B_2 B_3 B_4 B_5
$\mathcal{P}_2$
C_1 C_2 C_3 C_4 C_5 C_6 C_7


$\mathcal{P}_{-1}$
α
$\mathcal{P}_0$
A_1
A_2
A_3
$\mathcal{P}_1$
B_1
B_2
B_3
B_4
B_5
$\mathcal{P}_2$
C_1
C_2
C_3
C_4
C_5
C_6
C_7

Let $\lambda \in \mathcal{P}_d$.

$$\begin{aligned}\Gamma^\lambda &= \{\text{paths from } \alpha \text{ to } \lambda\} \\ &= \{\text{tableau of a skew shape with entries } 1, 2, \ldots, d.\}\end{aligned}$$

For $0 \leq i \leq d-1$, $T \in \Gamma^\lambda$.

$$s_i.T = \begin{cases}\text{unique other path which differs from } T \text{ at row } i \\ \star \quad \text{otherwise}\end{cases}$$

Fix $\lambda \in \mathcal{P}_d$, $f : \Gamma^\lambda \to \mathbb{C}$ such that

$$f(T)f(s_0.T) = \text{an explicit function in } \kappa_T(i)$$

The modules $\mathcal{D}_f^\lambda$

Recall $\mathcal{H}_d^p = \mathcal{B}_d/\sim$.

Define $\mathcal{D}_f^\lambda = \bigoplus_{T\in\Gamma^\lambda} \mathrm{Cl}_d\, v_T$ a free Cl_d-module.

Theorem (Z.)

$\mathcal{H}_d^p$ has an alternative presentation using generators $x_1, z_0, \dots, z_d, s_1, \dots, s_{d-1}, c_1, \dots, c_d$.

parity of v_T: depends on the zero-th edge in T.

Let

$$z_0.v_T = (\text{function in the zero-th edge in } T)v_T$$
$$z_i.v_T = \sqrt{c_T(i)(c_T(i)+1)}v_T \quad 1 \leq i \leq d$$

Recall $x_1 z_j = z_j x_1$, $j \geq 2$.
x_1 acts on

$$\langle v_T, \quad c_0 c_1 v_T, \quad c_0 v_{s_0.T}, \quad c_1 v_{s_0.T} \rangle$$

via an explicit matrix in terms of T.
s_i acts on

$$\langle v_T, \quad c_0 c_1 v_T, \quad v_{s_i.T}, \quad c_0 c_1 v_{s_i.T} \rangle$$

with an explicit matrix as in Hill-Kujawa-Sussan.

Theorem (Z.)

$\mathcal{D}_f^\lambda$ admits a well-defined action of $\mathcal{H}_d^p$ given as above. Furthermore, the action of x_1, s_i is uniquely determined by the action of $z_0, \ldots, z_d$, up to a choice of f.

Theorem (Z.)

$\mathcal{D}_f^\lambda$ is irreducible.
$\lambda \not\simeq \mu$: $\mathcal{D}^\lambda \neq \mathcal{D}^\mu$
$\mathcal{D}_f^\lambda$ is an infinite family of nonisomorphic modules.

Recall

$$\rho : \mathcal{H}_n^p \to \mathcal{Z}_d := \mathrm{End}_{\mathfrak{q}(n)}(L(\alpha) \otimes L(\beta) \otimes V^{\otimes d})$$

$\mathcal{L}^\lambda$: irreducible $\mathcal{Z}_d$-summands of $L(\alpha) \otimes L(\beta) \otimes V^{\otimes d}$

Theorem (Z.)

$\mathrm{Res}^{\mathcal{Z}_d}_{\rho(\mathcal{H}_d^p)} \mathcal{L}^\lambda \simeq \mathcal{D}_f^\lambda$ *for some* f.

Corollary (Z.)

$\mathrm{Res}^{\mathcal{Z}_d}_{\rho(\mathcal{H}_d^p)} \mathcal{L}^\lambda$ *is irreducible.*

$\rho(\mathcal{H}_d^p) \subset \mathcal{Z}_d$: “dense” subalgebra.