Two boundary centralizer algebras for $\mathfrak{q}(n)$

Jieru Zhu

University at Buffalo

$$A \quad \circlearrowright W \circlearrowleft \quad B$$

	A	W	B
Schur (1905)	$GL_n(\mathbb{C})$	$V^{\otimes d}$	S_n
Sergeev (1985)	$\mathfrak{q}_n(\mathbb{C})$	$V^{\otimes d}$	$\mathcal{S}_d$

Arakawa-Suzuki (1998)	$\mathfrak{sl}_n(\mathbb{C})$	$M \otimes V^{\otimes d}$	AHA
Hill-Kujawa-Sussan (2009)	$\mathfrak{q}_n(\mathbb{C})$	$M \otimes V^{\otimes d}$	AHCA

Daugherty (2010)	$\mathfrak{sl}_n(\mathbb{C})$	$M \otimes N \otimes V^{\otimes d}$	2 bndry AHA
our case	$\mathfrak{q}_n(\mathbb{C})$	$M \otimes N \otimes V^{\otimes d}$	2 bndry AHCA

Type Q Lie superalgebra

The Type Q Lie superalgebra

$$\mathfrak{q} = \mathfrak{q}(n) = \mathfrak{q}_{\overline{0}} \oplus \mathfrak{q}_{\overline{1}}$$
$$\mathfrak{q}_{\overline{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} |A \in \mathrm{Mat}_{n,n}(\mathbb{C}) \right\},$$
$$\mathfrak{q}_{\overline{1}} = \left\{ \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} |B \in \mathrm{Mat}_{n,n}(\mathbb{C}) \right\}.$$

$x \in \mathfrak{q}(n)_{\overline{x}}$: x homogeneous of degree $\overline{x}$.

Lie superbracket

$$[x, y] = xy - (-1)^{\overline{x}\cdot\overline{y}} yx$$

$V = \mathbb{C}^{2n}$: $\mathfrak{q}(n)$-module
action: matrix multiplication on the left.

Lie superbracket

$$[x, y] = xy - (-1)^{\overline{x}\cdot\overline{y}}yx$$

$V = \mathbb{C}^{2n}$: $\mathfrak{q}(n)$-module
action: matrix multiplication on the left.

M, N: $\mathfrak{q}(n)$-module $\Longrightarrow M \otimes N$: $\mathfrak{q}(n)$-module

$$\mathfrak{q}(n) \curvearrowright M \otimes N \otimes V^{\otimes d} \curvearrowleft ??$$

Degenerate two boundary affine Hecke-Clifford algebra

A superalgebra $S = S_{\overline{0}} \oplus S_{\overline{1}}$,

$$S_{\overline{i}} \cdot S_{\overline{j}} \subset S_{\overline{i+j}}, \qquad \overline{i}, \overline{j} \in \mathbb{Z}_2$$

Degenerate two boundary affine Hecke-Clifford algebra

A superalgebra $S = S_{\overline{0}} \oplus S_{\overline{1}}$,

$$S_{\overline{i}} \cdot S_{\overline{j}} \subset S_{\overline{i+j}}, \qquad \overline{i}, \overline{j} \in \mathbb{Z}_2$$

The degenerate two boundary affine Hecke-Clifford algebra $\mathcal{B}_d$: generated by
1) polynomial rings $\mathbb{C}[\tilde{x}_1, \ldots, \tilde{x}_d]$, $\mathbb{C}[\tilde{y}_1, \ldots, \tilde{y}_d]$, $\mathbb{C}[\tilde{z}_0, \tilde{z}_1, \ldots, \tilde{z}_d]$ (polynomial generators: odd)

2) even simple transpositions: $s_1, \ldots, s_{d-1}$, subject to S_d-relation

3) odd Clifford generators: $c_1, \ldots, c_d$, subject to $c_i^2 = -1$, $c_i c_j = -c_j c_i (i \neq j)$

the “Hecke” relations

$$s_i\tilde{x}_i = \tilde{x}_{i+1}s_{i+1} + (c_i - c_{i+1})$$

A Casimir element

$$\Omega \in \mathfrak{q}(n) \otimes \mathfrak{q}(n)$$

$\Omega_{M,N}$: its action on $M \otimes N$

the “Hecke” relations

$$s_i\tilde{x}_i = \tilde{x}_{i+1}s_{i+1} + (c_i - c_{i+1})$$

A Casimir element

$$\Omega \in \mathfrak{q}(n) \otimes \mathfrak{q}(n)$$

$\Omega_{M,N}$: its action on $M \otimes N$

$V = \mathbb{C}^{2n}$: simple $\mathfrak{q}(n)$-module

$$c = \begin{bmatrix} 0 & -1_n \\ 1_n & 0 \end{bmatrix} \in \mathrm{End}_{\mathfrak{q}(n)}(V)$$

Main Result

Theorem (Z.)

Let M, N be arbitrary $\mathfrak{q}(n)$-modules. There is a superalgebra homomorphism:

$$\mathcal{B}_d \to \mathrm{End}_{\mathfrak{q}(n)}(M \otimes N \otimes V^{\otimes d})$$

$$\tilde{x}_i \mapsto \Omega_{M\otimes V^{i-1},i} = \Omega_{M,i} + \Omega_{1,i} + \cdots + \Omega_{i-1,i}$$

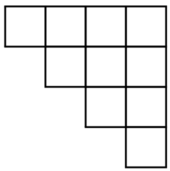
$$\tilde{y}_i \mapsto \Omega_{N\otimes V^{i-1},i} \quad \tilde{z}_i \mapsto \Omega_{M\otimes N\otimes V^{i-1},i} \quad \tilde{z}_0 \mapsto \Omega_{M,N}$$

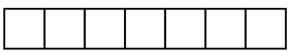
s_i: signed swap

c_i: the odd map on the i-th factor.

Irreducibles Polynomial representations of $\mathfrak{q}(n)$: $\{L(\lambda)\}$

λ "shifted Young diagram"

Let $\alpha =$  (staircase of height n)

$\beta =$  (single row with p boxes)

$$\text{content of } \square = \text{column of } \square - \text{row of } \square$$

Quotients of the Hecke-Clifford algebra

$\mathcal{H}_d^p = \mathcal{B}_d / \sim$ under further relations

$$(\tilde{x_1})^2 = n(n+1)$$
$$(\tilde{y_1})^2((\tilde{y_1})^2 - p(p+1)) = 0$$

Quotients of the Hecke-Clifford algebra

$\mathcal{H}_d^p = \mathcal{B}_d / \sim$ under further relations

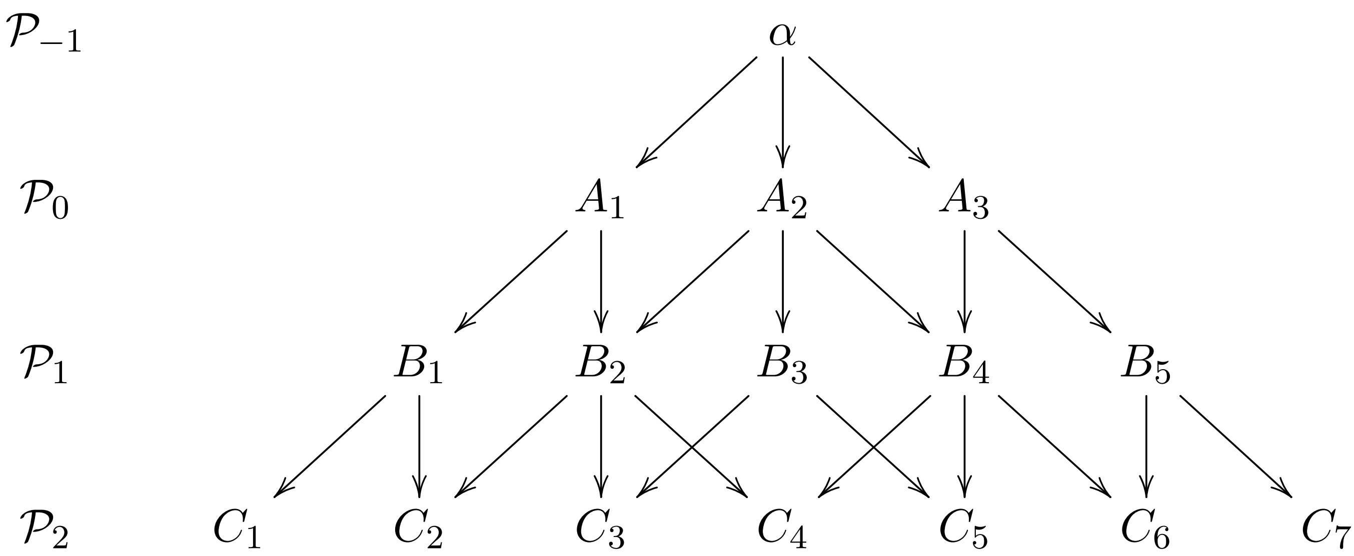
$$(\tilde{x_1})^2 = n(n+1)$$
$$(\tilde{y_1})^2((\tilde{y_1})^2 - p(p+1)) = 0$$

Theorem

The action of $\mathcal{B}_d$ factors through $\mathcal{H}_n^p$ and induces further a homomorphism of superalgebras

$$\mathcal{H}_n^p \to \mathrm{End}_{\mathfrak{q}(n)}(L(\alpha) \otimes L(\beta) \otimes V^{\otimes d})$$

The Bratteli graph



$$L(\lambda) \otimes L(\mu) \simeq \bigoplus_{\gamma} L(\gamma)^{\oplus c^{\gamma}_{\lambda,\mu}}$$

The Bratteli graph: a directed graph
Vertices $\cup_{i=-1}^{\infty} \mathcal{P}_i$

$$\mathcal{P}_{-1} = \{\alpha\}$$
$$\mathcal{P}_0 = \{\gamma \mid L(\gamma) \text{ is a summand of } L(\alpha) \otimes L(\beta)\}$$

for every $\lambda \in \mathcal{P}_0$, $\alpha \to \lambda$, $L(\lambda)$: multiplicity 2

$$L(\lambda) \otimes L(\mu) \simeq \bigoplus_{\gamma} L(\gamma)^{\oplus c^{\gamma}_{\lambda,\mu}}$$

The Bratteli graph: a directed graph
Vertices $\cup_{i=-1}^{\infty} \mathcal{P}_i$

$$\mathcal{P}_{-1} = \{\alpha\}$$
$$\mathcal{P}_0 = \{\gamma \mid L(\gamma) \text{ is a summand of } L(\alpha) \otimes L(\beta)\}$$

for every $\lambda \in \mathcal{P}_0$, $\alpha \to \lambda$, $L(\lambda)$: multiplicity 2

$$\mathcal{P}_1 = \{\gamma \mid L(\gamma) \text{ is a summand of } L(\lambda) \otimes V, \quad \exists \lambda \in \mathcal{P}_0\}$$

$L(\mu) \subset L(\lambda) \otimes V \iff \mu$ is obtained by adding a box to λ,

$\lambda \to \mu$ (Pieri Rule)

$$\mathcal{P}_i = \{\gamma \ \mid \ L(\gamma) \text{ is a summand of } L(\lambda) \otimes V, \quad \exists \lambda \in \mathcal{P}_{i-1}\}$$

Let $\lambda \in \mathcal{P}_d$.

$$\Gamma^\lambda = \{\text{paths from } \alpha \text{ to } \lambda\}$$

$$\mathcal{P}_i = \{\gamma \ \mid \ L(\gamma) \text{ is a summand of } L(\lambda) \otimes V, \quad \exists \lambda \in \mathcal{P}_{i-1}\}$$

Let $\lambda \in \mathcal{P}_d$.

$$\Gamma^\lambda = \{\text{paths from } \alpha \text{ to } \lambda\}$$

$\{s_0, s_1, \ldots, s_{d-1}\} \curvearrowright \Gamma^\lambda \cup \{\star\}$:

$s_i.T =$ the path by reversing the added boxes i and $i+1$
$(1 \leq i \leq d-1)$

Otherwise $s_i.T = \star$.

Combinatorial construction

Let $z_i = \tilde{z}_i c_i$. A $\mathcal{H}_d^p$-module is *calibrated* if it admits a basis on which z_i acts semisimply.

Define $\mathcal{D}_f^\lambda = \bigoplus_{T \in \Gamma^\lambda} \mathrm{Cl}_d\, v_T$ a free Cl_d-module.

Fix $\lambda \in \mathcal{P}_d$, $f : \Gamma^\lambda \to \mathbb{C}$ subject to an explicit condition.

Combinatorial construction

Let $z_i = \tilde{z}_i c_i$. A $\mathcal{H}_d^p$-module is *calibrated* if it admits a basis on which z_i acts semisimply.

Define $\mathcal{D}_f^\lambda = \bigoplus_{T \in \Gamma^\lambda} \mathrm{Cl}_d \, v_T$ a free Cl_d-module.

Fix $\lambda \in \mathcal{P}_d$, $f : \Gamma^\lambda \to \mathbb{C}$ subject to an explicit condition.

$\mathcal{H}_d^p$-module structure:

1) $z_i . v_T = \kappa_T(i) v_T$

$$\kappa_T(i) = \sqrt{c(i)\big(c(i)+1\big)} v_T$$

$$c(i) = \text{content of } i\text{-th added box}$$

2) s_i acts on $\langle v_T, \quad c_0c_1v_T, \quad v_{s_i.T}, \quad c_0c_1v_{s_i.T}\rangle$ via

$$\begin{bmatrix} -\frac{1}{\kappa_T(i)-\kappa_T(i+1)} & * & * & * \\ \frac{1}{\kappa_T(i)+\kappa_T(i+1)} & * & * & * \\ \sqrt{1-\frac{1}{(\kappa_T(i)+\kappa_T(i+1))^2}-\frac{1}{(\kappa_T(i)-\kappa_T(i+1))^2}} & * & * & * \\ 0 & * & * & * \end{bmatrix}$$

where $v_\star = 0$.

3) Let $N_0 = n(n+1)$, x_1 acts on $\langle v_T, \;\; c_0c_1v_T, \;\; c_0v_{s_0.T}, \;\; c_1v_{s_0.T}\rangle$ via

$$\begin{bmatrix} \frac{\kappa_T(1)}{\kappa_T(0)^2+p\kappa_T(1)^2}\big(pN_0+\kappa_T^2(0)\big) & * & * & * \\ \frac{\kappa_T(0)}{\kappa_T(0)^2+p\kappa_T(1)^2}\big(-N_0+\kappa_T^2(1)\big) & * & * & * \\ f(T)\big(\kappa_T(0)-\kappa_{s_0.T}(0)\big) & * & * & * \\ f(T)\big(\kappa_T(1)+\kappa_{s_0.T}(1)\big) & * & * & * \end{bmatrix}$$

where $v_\star = 0$

Main results

Theorem (Z.)

$\mathcal{D}_f^\lambda$ admits a well-defined action of $\mathcal{H}_d^p$ given as above.

Theorem (Z.)

$\mathcal{D}_f^\lambda$ is irreducible.

Theorem (Z.)

The action of x_1, s_i is uniquely determined by the action of $z_0, \ldots, z_d$, up to a choice of f.

Recovering calibrated modules

Let

$$\mathcal{Z}_d := \mathrm{End}_{\mathfrak{q}(n)}(L(\alpha) \otimes L(\beta) \otimes V^{\otimes d})$$

$L(\alpha) \otimes L(\beta) \otimes V^{\otimes d}$: $(\mathfrak{q}(n), \mathcal{Z}_d)$-bimodule

$\mathcal{L}^{\lambda}$: irreducible $\mathcal{Z}_d$-summands of $L(\alpha) \otimes L(\beta) \otimes V^{\otimes d}$

Recall $\rho : \mathcal{H}_d^p \to \mathcal{Z}_d$.

Recovering calibrated modules

Let

$$\mathcal{Z}_d := \mathrm{End}_{\mathfrak{q}(n)}(L(\alpha) \otimes L(\beta) \otimes V^{\otimes d})$$

$L(\alpha) \otimes L(\beta) \otimes V^{\otimes d}$: $(\mathfrak{q}(n), \mathcal{Z}_d)$-bimodule

$\mathcal{L}^\lambda$: irreducible $\mathcal{Z}_d$-summands of $L(\alpha) \otimes L(\beta) \otimes V^{\otimes d}$

Recall $\rho : \mathcal{H}_d^p \to \mathcal{Z}_d$.

Theorem (Z.)

$\mathrm{Res}^{\mathcal{Z}_d}_{\rho(\mathcal{H}_d^p)} \mathcal{L}^\lambda \simeq \mathcal{D}_f^\lambda$ *for some* f.