Transitioning Between The Tableaux Bases and The Web Bases for The Specht Modules

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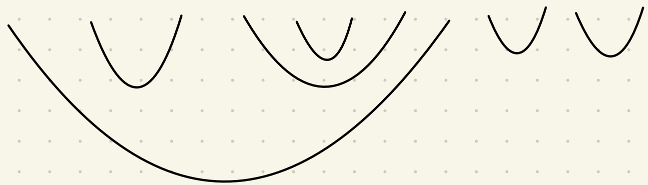
AMS sectional, 2021

COMBINATORIAL CONSTRUCTION

$\lambda = (d, d)$, $n = 2$, S_{2d}.

D^λ : $\mathbb{C}$- v.s. with basis: all cup diagrams

e.g.



(Non-crossing matchings) with 2d boundary pts.

S_{2d}-action:

$s_i \cdot$

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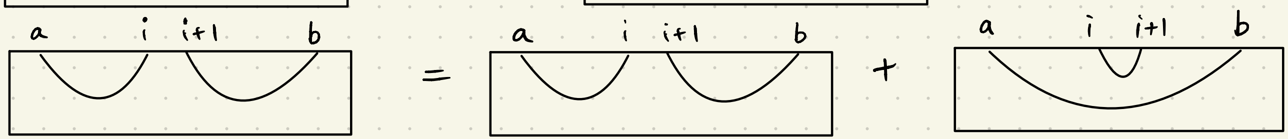
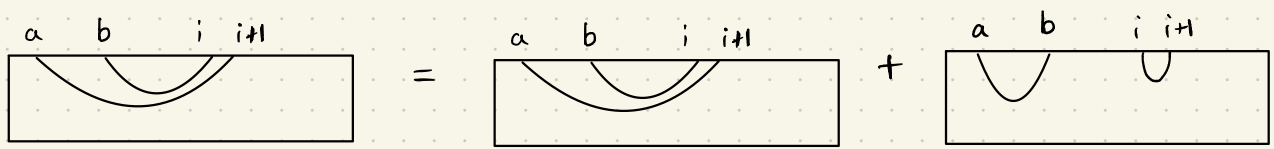
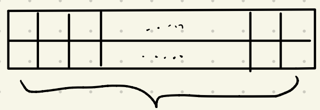


$s_i \cdot$



COMBINATORIAL CONSTRUCTION Ver. 2

Young diagram of shape $\lambda = (d, d)$



d boxes

Young Tableau of shape λ with entries $1, 2, \cdots, 2d$

e.g. $d = 5$

6	1	7	8	5
2	10	4	3	9

STANDARD IF:

e.g.

1	2	4	7	8
3	5	6	9	10

entries increase $\rightarrow$ & $\downarrow$

Tabloid of shape λ

e.g.

6 1 7 8 5
2 10 4 3 9

=

7 1 8 6 5
3 4 9 2 10

Polytabloid Basis (v_T: Polytabloid)

$\begin{cases} S_d \curvearrowright \text{tabloid by permuting the entries} \\ v_T = \sum_{\sigma} \operatorname{sgn}(\sigma).(\text{tabloid } T) \quad \text{sum over all } \sigma\text{: column stablizer of } T \end{cases}$

$\bar{D}^\lambda$: $\mathbb{C}$- v.s spanned by $\{ v_T \mid T\text{: all Young Tab.}\}$ $\quad S_{2d} \curvearrowright \bar{D}^\lambda$

Theorem $\bar{D}^\lambda$ has a basis $\{ v_T \mid T\text{: standard Young Tab}\}$

$\{v_T\}$ satisfy the Garnir Relations

Remark $\bar{D}^\lambda$ is irreducible :

$$s_i \cdot v_{\begin{array}{|ccc|}\hline \quad & \quad & \boxed{i{+}1} \\ \hline \quad & \boxed{i} & \quad \\ \hline\end{array}} = v_{\begin{array}{|ccc|}\hline \quad & \quad & \boxed{i{+}1} \\ \hline \quad & \boxed{i} & \quad \\ \hline\end{array}} + v_{\begin{array}{|ccc|}\hline \quad & \quad & \boxed{i} \\ \hline \quad & \boxed{i{+}1} & \quad \\ \hline\end{array}}$$

The Tableaux Graph $(n=2,\ d=3)$

(Russell-Tymoczko 2020)

$$
\begin{tikzcd}
& w_0 = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline \end{array} \arrow[dl, "s_2"'] \arrow[dr, "s_4"] & \\
w_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \arrow[dr, "s_4"'] & & w_2 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array} \arrow[dl, "s_2"] \\
& w_3 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array} \arrow[d, "s_3"] & \\
& w_4 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array} &
\end{tikzcd}
$$

Def

$$T_0 = \begin{array}{|c|c|c|c|}\hline 1 & 3 & \cdots & 2d-1 \\ \hline 2 & 4 & \cdots & 2d \\ \hline \end{array}$$

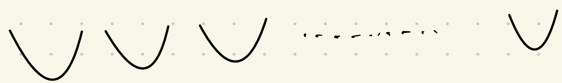
$R \leq S$

$\iff \exists$ directed path from R to S

Module Isomorphism

Th $\overline{D}^\lambda \simeq D^\lambda$ (Specht module corresponding to cycle type λ)

and up to a scalar,

$v_{T_0} =$  $=: w_0$

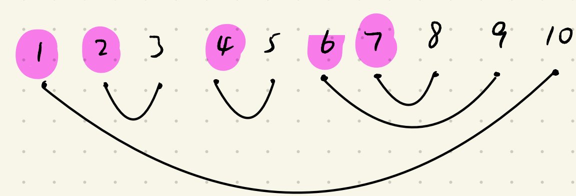
In fact, $\dim \overline{D}^\lambda = \dim D^\lambda = d^{th}$ Catalan number

$\phi : \{\text{standard Young Tableaux}\} \xrightarrow{\sim} \{\text{cup diagrams}\}$

e.g.

1	2	4	6	7
3	5	8	9	10

$\mapsto$



"Catalan Identification"

Unitriangularity Result

Transfer "$\leq$" on $\{$standard Young Tab$\}$ to $\{$cup diag$\}$

Th Russell – Tymoczko (2020)

$$v_T = \phi(T) + (\text{ terms } \leq \phi(T))$$

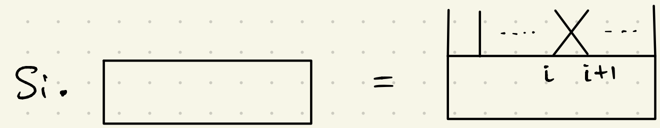
Conjecture (Russell – Tymoczko) POSITIVITY

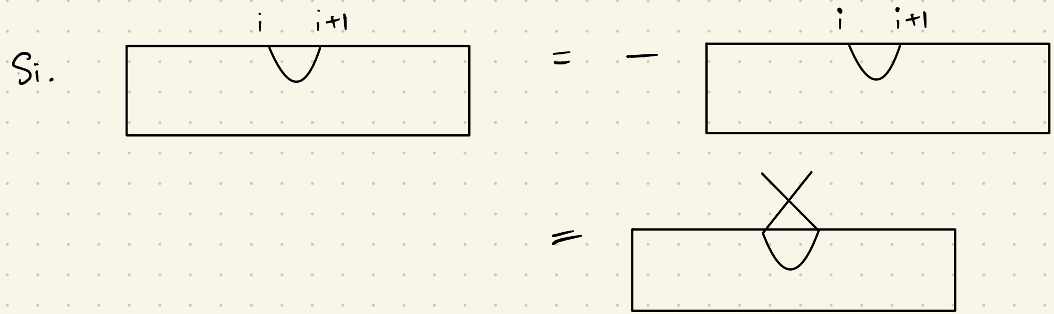
- Lower terms have positive coeff
- all lower terms occur

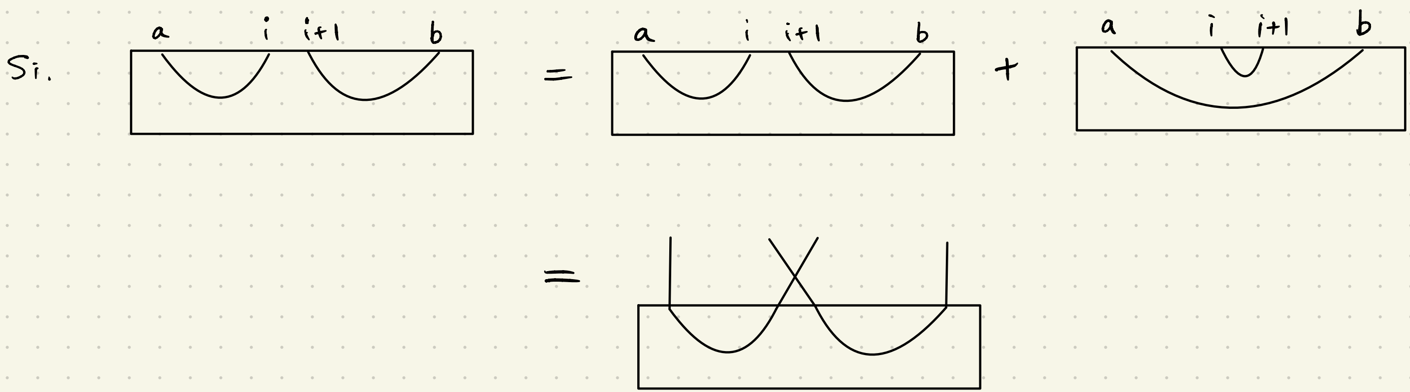
Th (Rhoades 2020) Nonnegativity

- all lower terms have NONNEGATIVE coefficients

Nonnegativity Result, reformulated



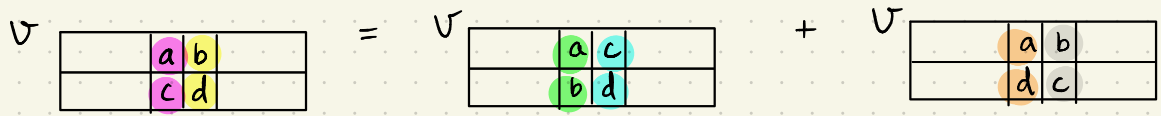




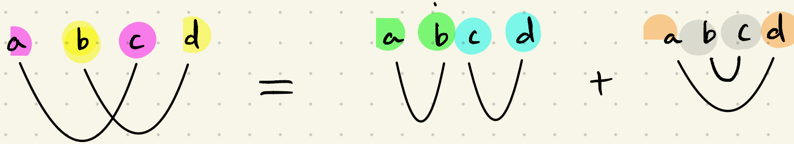
Relations are equivalent

Tableaux

(Garnir Relations)

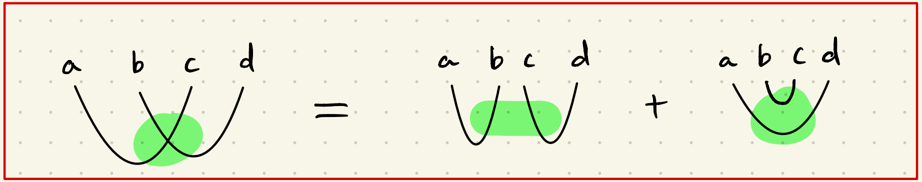


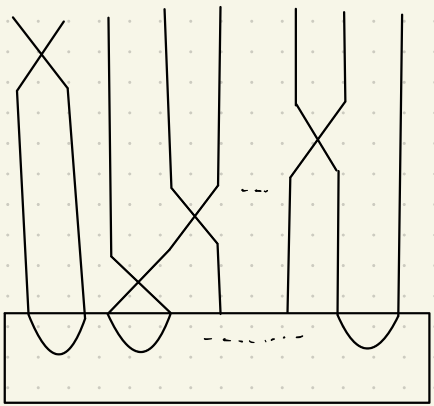
diagrams with crossings



(Proposed Relation)

Braid on top of Wo





Relation is "GLOBAL"

Note : Skein relation is LOCAL

Replace all  by − 

Then expands positively !

A web for A Tableau

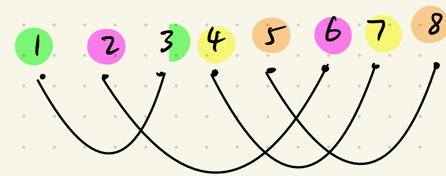
Prop. (Im-Z. 2021)

In $\bar{D}^\lambda \simeq D^\lambda$ under the map

e.g. U

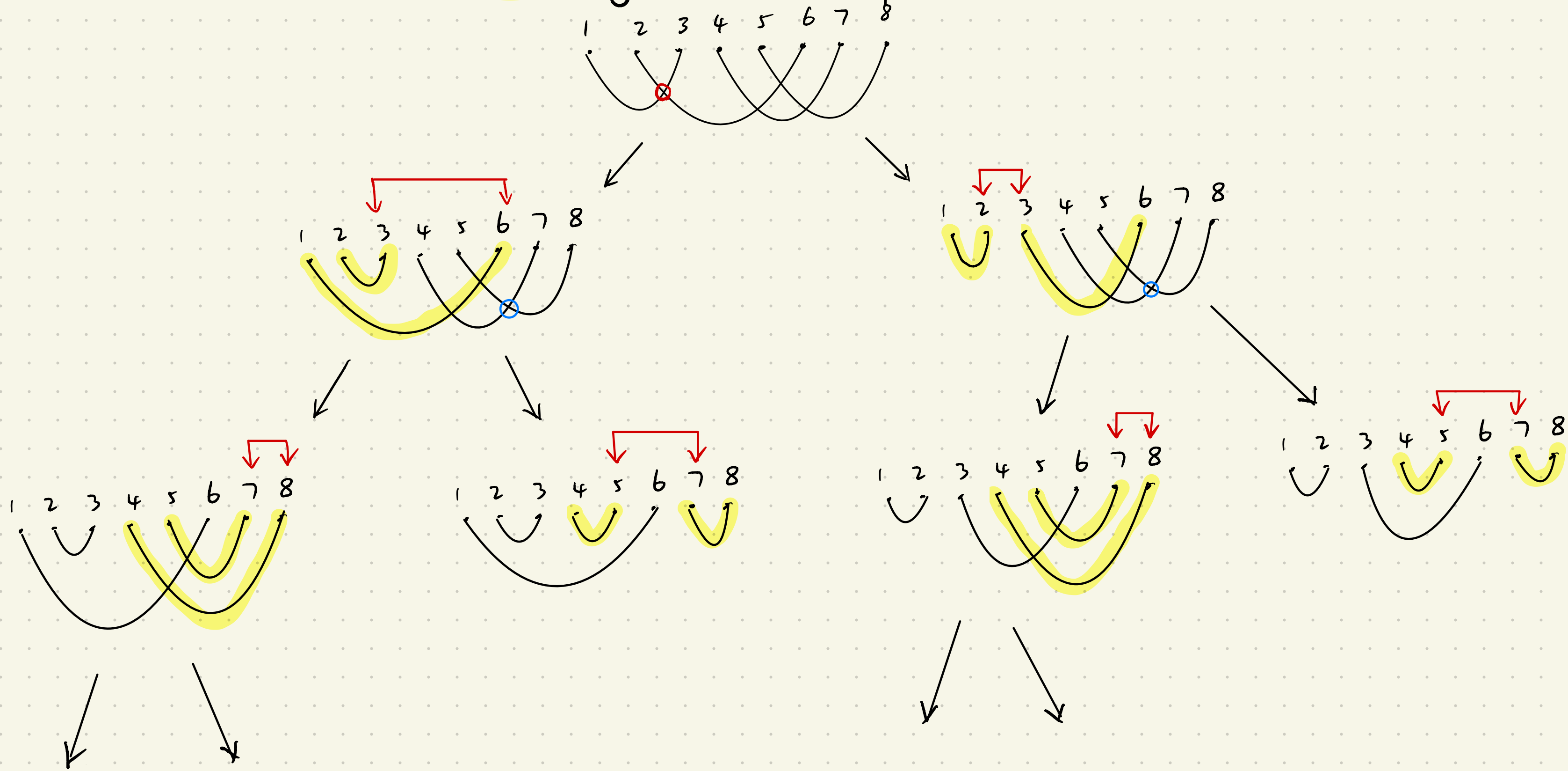
1	2	4	5
3	6	7	8

$\mapsto$



"column identification"

Crossing Resolving Graph



...... until no crossings remain!

Existence of Path

Observation 1) "Repetitions" coeff of a cup diag D = # path from the top to D

2) D occurs in the expansion $\Leftrightarrow$ $\exists$ a path to D

Remark. This graph is NOT uniquely defined

Propostion (Im - 2.) POSITIVITY $\qquad \phi$: Catalan identification

$$V_T = \phi(T) + \text{lower Terms}$$

$\phi(S)$ occurs on RHS $\Leftrightarrow$ $S \leq T$

Proof (by example)

S_5

$S \leq T$

$T =$

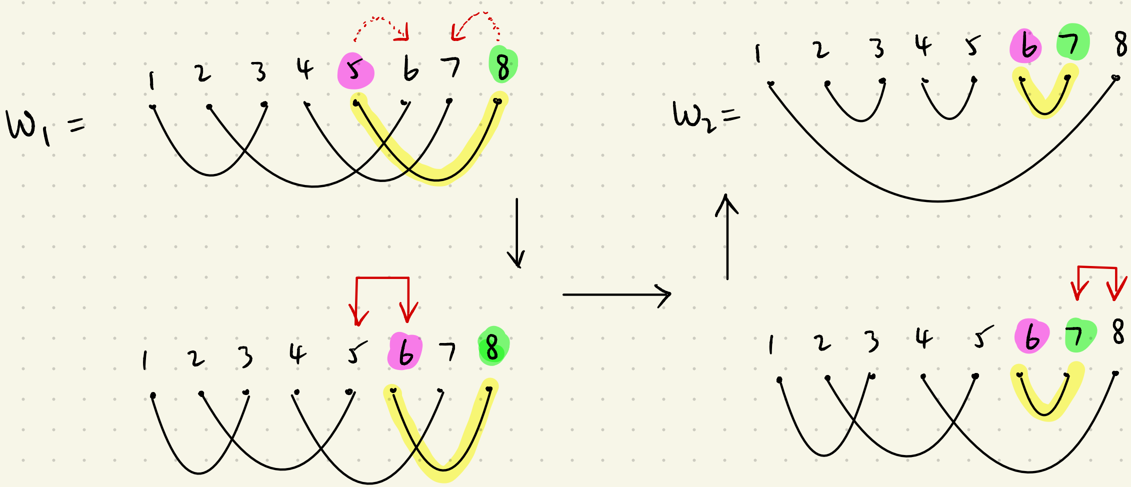
1	2	4	5
3	6	7	8

$S =$

1	2	4	6
3	5	7	8

GOAL

- W_2 occurs in the expansion of V_T
- $\exists$ a path from W_1 to W_2



Induction

$T =$

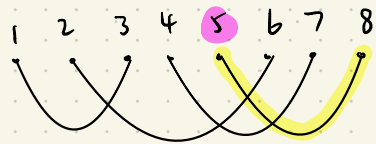
1	2	4	5
3	6	7	8

$\xleftarrow{s_5}$

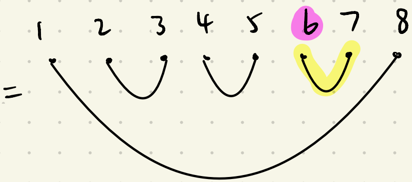
$S =$

1	2	4	6
3	5	7	8

$w_1 =$



$w_2 =$



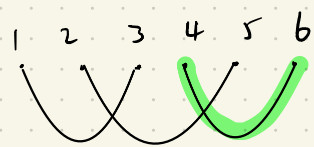
$T' =$

1	2	4
3	5	6

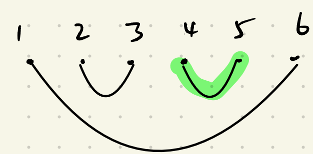
$S' =$

1	2	4
3	5	6

$w_1' =$



$w_2' =$



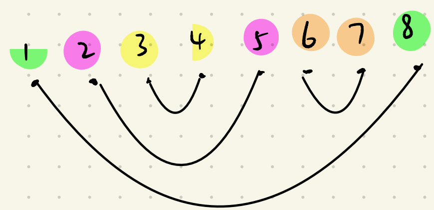
The Inverse Map

Prop (Im-Z.)

If W: cup diag, then $W = V_T$, where

T: the (nonstandard) Young Tab given by the

"column identification"

e.g. $W =$ 

$T =$

1	2	3	6
8	5	4	7

Conjecture W expands into V_T's with coeff $\pm 1, 0$

Ongoing Work

- Positivity for $\mathfrak{sl}_3$- webs
- Quantum case
- Connection to K-L polynomials.

THANK YOU !