

Transitioning Between The Tableaux Bases

and The web Bases for The Specht Modules

Mee Seong Im , Jieru Zhu

IAS Summers Collaborators Program 2019

AMS sectional, March 2021

COMBINATORIAL CONSTRUCTION

$\lambda = (d, d)$, $n=2$, S_{2d} .

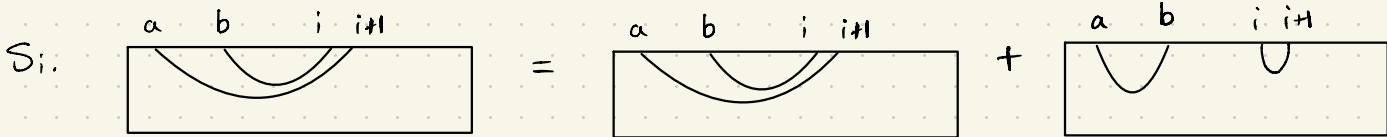
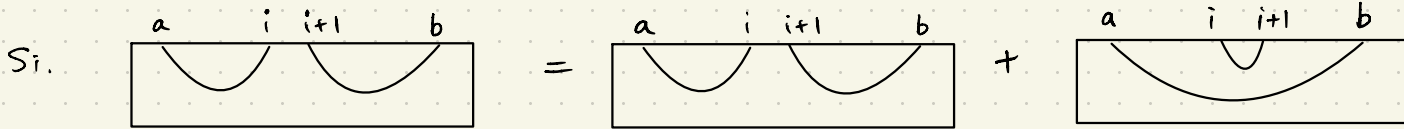
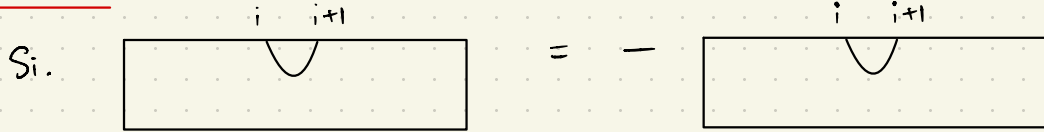
D^λ : \mathbb{C} -v.s. with basis: all cup diagrams

e.g.



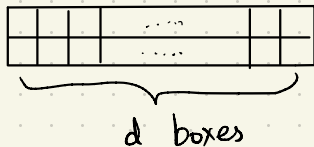
(Non-crossing matchings)
with $2d$ boundary pts.

S_{2d} -action:



COMBINATORIAL CONSTRUCTION Ver. 2

Young diagram of shape $\lambda = (d, d)$



Young Tableau of shape λ with entries $1, 2, \dots, 2d$

e.g. $d=5$

6	1	7	8	5
2	10	4	3	9

STANDARD IF:

e.g.

1	2	4	7	8
3	5	6	9	10

entries increase \rightarrow & \downarrow

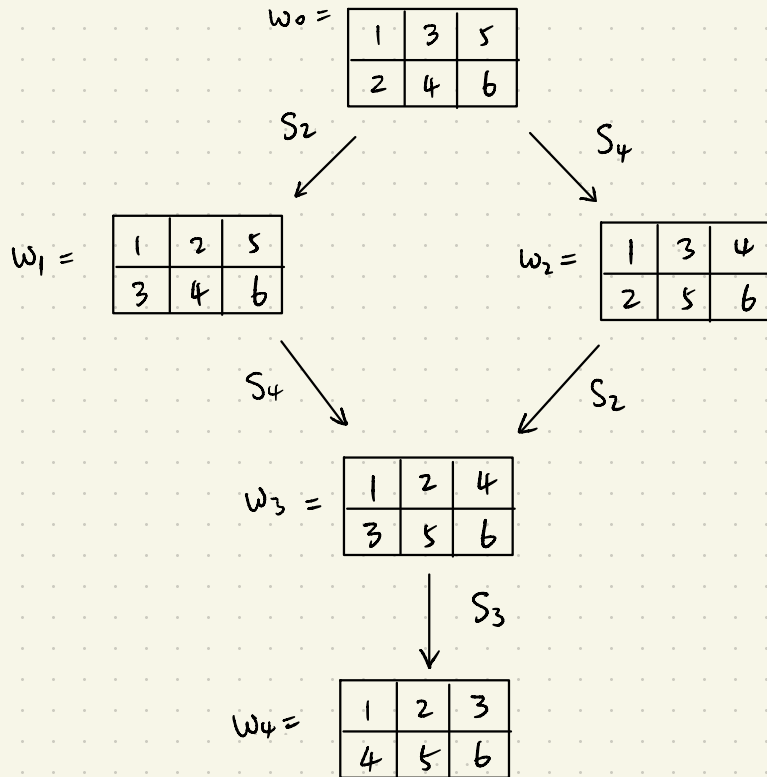
Tabloid of shape λ

e.g.

$$\begin{array}{|c|c|c|c|c|} \hline 6 & 1 & 7 & 8 & 5 \\ \hline 2 & 10 & 4 & 3 & 9 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \hline 7 & 1 & 8 & 6 & 5 \\ \hline 3 & 4 & 9 & 2 & 10 \\ \hline \end{array}$$

The Tableaux Graph $(n=2, d=3)$

(Russell-Tymoczko 2020)



Def

$$T_0 = \begin{array}{|c|c|c|c|} \hline 1 & 3 & \dots & 2d-1 \\ \hline 2 & 4 & \dots & 2d \\ \hline \end{array}$$

$$R \leq S$$

$\Leftrightarrow \exists$ directed path from R to S

Module Isomorphism

Th $\bar{D}^\lambda \cong D^\lambda$ (Specht module corresponding to cycle type λ)
and up to a scalar,

$$V_{T_0} = \cup \cup \cup \dots \cup =: W_0$$

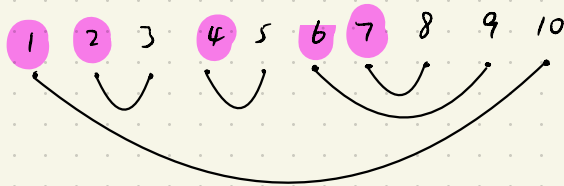
In fact, $\dim \bar{D}^\lambda = \dim D^\lambda = d^{\text{th}}$ Catalan number

$$\phi: \{ \text{standard Young Tableaux} \} \xrightarrow{\sim} \{ \text{cup diagrams} \}$$

e.g.

1	2	4	6	7
3	5	8	9	10

\mapsto



"Catalan Identification"

Unitriangularity Result

Transfer " \leq " on $\{\text{standard Young Tab}\}$ to $\{\text{cup diag}\}$

Th Russell - Tymoczko (2020)

$$v_T = \phi(T) + (\text{terms } \leq \phi(T))$$

Conjecture (Russell - Tymoczko) POSITIVITY

- Lower terms have positive coeff
- all lower terms occur

Th (Rhoades 2020) Nonnegativity

- all lower terms have NONNEGATIVE coefficients

Nonnegativity Result, reformulated

$$S_i. \quad \boxed{} = \frac{\boxed{\dots} \times \boxed{\dots}}{\boxed{i \quad i+1}}$$

$$S_i. \quad \boxed{} \quad \begin{matrix} i & i+1 \\ \cup \end{matrix} = - \quad \boxed{} \quad \begin{matrix} i & i+1 \\ \cup \end{matrix}$$

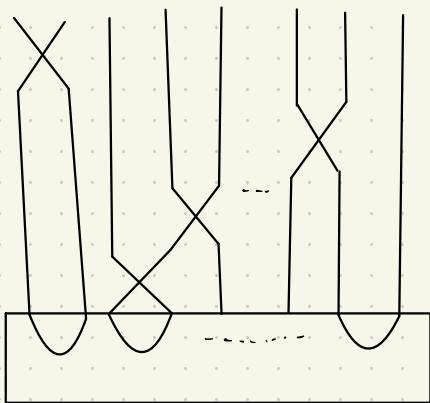
$$= \boxed{} \quad \begin{matrix} \times \\ \cup \end{matrix}$$

$$S_i. \quad \boxed{} \quad \begin{matrix} a & i & i+1 & b \\ \cup & \cup \end{matrix} = \boxed{} \quad \begin{matrix} a & i & i+1 & b \\ \cup & \cup \end{matrix} + \boxed{} \quad \begin{matrix} a & i & i+1 & b \\ \cup \end{matrix}$$

$$= \boxed{} \quad \begin{matrix} \times \\ \cup & \cup \end{matrix}$$

Braid on top of W_0

$$\begin{array}{cccc} a & b & c & d \\ \cup & \cup & \cup & \cup \\ \cup & \cup & \cup & \cup \\ \cup & \cup & \cup & \cup \end{array} = \begin{array}{cccc} a & b & c & d \\ \cup & \cup & \cup & \cup \\ \cup & \cup & \cup & \cup \\ \cup & \cup & \cup & \cup \end{array} + \begin{array}{cccc} a & b & c & d \\ \cup & \cup & \cup & \cup \\ \cup & \cup & \cup & \cup \\ \cup & \cup & \cup & \cup \end{array}$$



Relation is "GLOBAL"

Note: Skein relation is LOCAL

Replace all  by $-\cup$

Then expands positively!

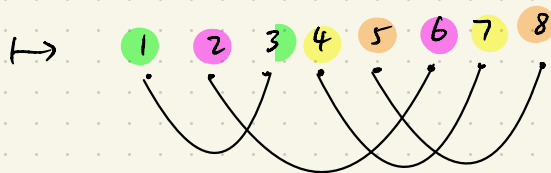
A web for A Tableau

Prep. (Im-2. 2021)

In $\bar{D}^\lambda \simeq D^\lambda$ under the map

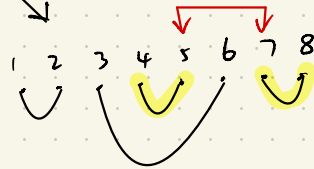
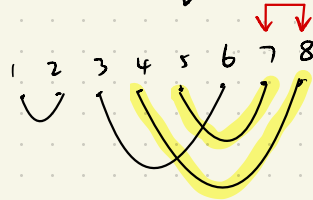
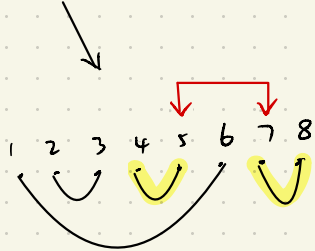
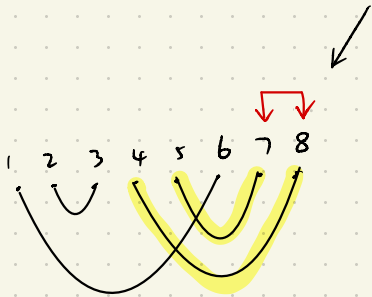
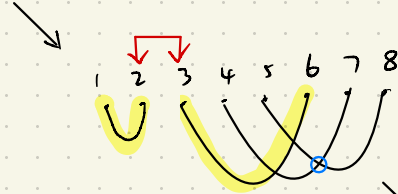
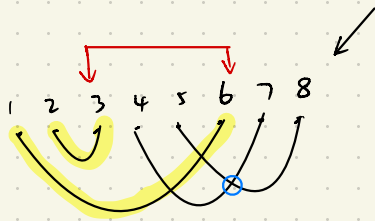
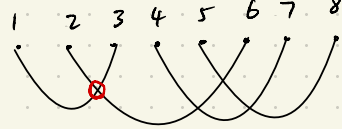
e.g. \bar{v}

1	2	4	5
3	6	7	8



"Column
identification"

Crossing Resolving Graph



..... until no crossings remain!

Existence of Path

Observation

- 1) "Repetitions" coeff of a cup diag $D = \#$ path from the top to D
- 2) D occurs in the expansion $\Leftrightarrow \exists$ a path to D

Remark

This graph is NOT uniquely defined

Proposition

(Im-2.) POSITIVITY

ϕ : Catalan identification

$$V_T = \phi(T) + \text{lower Terms}$$

$$\phi(S) \text{ occurs on RHS} \Leftrightarrow S \leq T$$

Proof (by example)

S_5

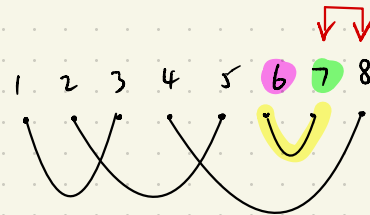
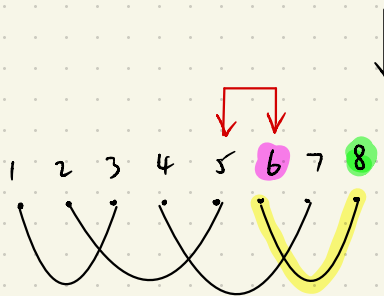
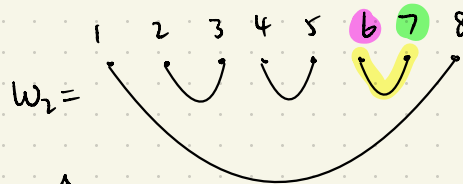
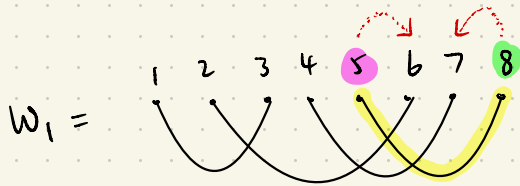
$S \leq T$

$T =$

1	2	4	5
3	6	7	8

$S =$

1	2	4	6
3	5	7	8



- GOAL
- w_2 occurs in the expansion of v_T
 - \exists a path from w_1 to w_2

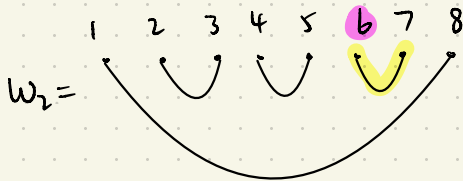
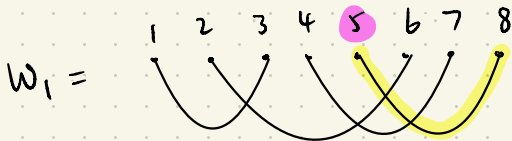
Induction

 $T =$

1	2	4	5
3	6	7	8

 $\xleftarrow{S_5}$ $S =$

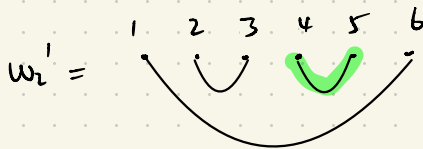
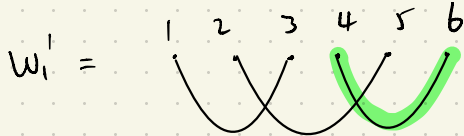
1	2	4	6
3	5	7	8

 $T' =$

1	2	4
3	5	6

 $S' =$

1	2	4
3	5	6

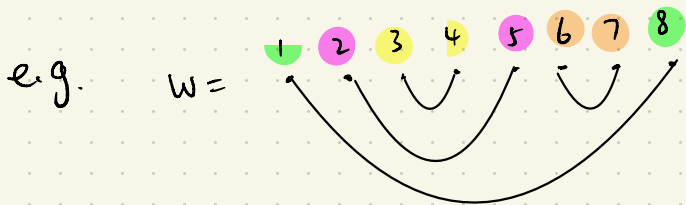


The Inverse Map

Prop (Im-Z.)

If W : cup diag, then $W = V_T$, where

T : the (nonstandard) Young Tab given by the
"column identification"



$T =$

1	2	3	6
8	5	4	7

Conjecture W expands into V_T 's with coeff $\pm 1, 0$

Ongoing Work

- Positivity for Sl_3 -webs
- Quantum case
- Connection to $K-L$ polynomials.

THANK YOU !

