Double - Centralizer Properties for The Drinfeld Double of The Taft Algebra

– Georgia Benkart, Rekha Biswal,
Ellen Kirkman, Van Nguyen, Jieru Zhu*

Women in Commutative Algebra and Representation Theory

"WINART" workshop

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The Big Picture

Temperley-Lieb algebra $\curvearrowright$ $V^{\otimes k}$ $\curvearrowleft$ Drinfeld Double of the Taft algebra

motivation:

$(\mathbb{C}S_d, \ \mathfrak{gl}_n(\mathbb{C}))$ - Schur-Weyl duality

The Drinfeld double of the Taft alg

over $\Bbbk$

Setup: $\Bbbk = \overline{\Bbbk}$. $\quad \operatorname{char} \Bbbk = 0 \quad q \in \Bbbk$

q: primative n-th root of unity

Definition: $D_n := \langle a, b, c, d \mid \sim \rangle$

relations:

$$\left[\begin{array}{ll} ab = q^{-1} ba & ac = q^{-1} ca \\ db = q\, bd & dc = q\, cd \\ da - qad = 1 - bc & bc = cb \\ a^n = d^n = 0 & c^n = d^n = 1 \end{array}\right]$$

$(D_n, \Delta, S, \varepsilon)$ is a Hopf algebra

b, c : grouplike

Remark : D_n is nonsemisimple

(Chen) D_n has a complete list of

simples $\quad V(\ell, r) \qquad 1 \leq \ell \leq n$

projectives $\quad P(\ell, r) \qquad 0 \leq r \leq n-1$

Fusion Rules

$$V(\ell, r) \otimes V(1, r') \simeq V(\ell, r+r')$$

etc...

☒ (Chen) D_n is Quasitriangular, i.e.

$\exists\ R \in D_n \otimes D_n$ satisfying axioms by Drinfeld

1) if M, N : D_n-mod

$R \in \mathrm{End}(M \otimes N)$ is D_n-linear

2) V : D_n-mod

$$R_i \curvearrowright V^{\otimes k} : \quad V \otimes \cdots \underset{\curvearrowleft R \nearrow}{\overset{i}{V} \otimes \overset{i+1}{V}} \otimes \cdots \otimes V$$

Braid Group Action

(Ram - Leduc '97) $\check{R}_i = \sigma R_i$ σ : swap

$$\boxed{\text{then } \check{R}_i \text{ satisfy the Braid Relations.}}$$

Setup: $q^{1/2}$ = root of the eq $x^2 - q = 0$

(well-def up to a sign)

$$\xi = -(q^{1/2} + q^{-1/2})$$

The Temperley-Lieb Algebra

$TL_k(\xi)$: generators $t_1, t_2, \cdots, t_{k-1}$

& relations

Braid relations

$$t_i t_j = t_j t_i \qquad |i-j|>1$$

$$t_i t_{i+1} t_i - t_i = t_{i+1} t_i t_{i+1} - t_i$$

Type A Hecke relations

$$t_i^2 = \xi t_i$$

$$t_i t_{i+1} t_i - t_i = t_{i+1} t_i t_{i+1} - t_i = 0$$

Main Result

Theorem (BBKNZ)

$\exists$ well-defined D_n-linear action of $TL_k(\delta)$ on $V(2,r)^{\otimes k}$

i.e.

$$\boxed{\begin{aligned} TL_k(\delta) &\longrightarrow \operatorname{End}_{D_n}\left(V(2,r)^{\otimes k}\right) \\ t_i &\longmapsto q^{1/2}\left(q^{-r(r+1)}\check{R}_i - 1\right) \end{aligned}}$$

The RIBBON Structure on D_n

(Kauffman-Radford '93)

$$\boxed{D_n \text{ is ribbon} \iff n : \text{odd}}$$

i.e., $\exists \, v$ s.t. $v^2 = u$

u : "naive" central element obtained from R

Digression :

D_n-mod : The category of D_n-modules is Ribbon (n : odd)

v = 

Theorem (BBKNZ)

$$\boxed{v = u\,(bc)^{\frac{n-1}{2}}}$$

Main Result

Fix $V(2, \frac{n-1}{2})$ Note: $V(2, \frac{n-1}{2}) \simeq V^*(2, \frac{n-1}{2})$

Theorem (BBKNZ)

$$\boxed{TL_k(\delta) \longrightarrow \mathrm{End}_{D_n}\left(V(2, \tfrac{n-1}{2})^{\otimes k} \right)}$$

a) is ALWAYS injective

b) is surjective ONLY WHEN $k \leq 2(n-1)$

($\Longleftrightarrow$)

Injectivity Statement

TFAE:

1) $\varphi : TL_k \longrightarrow \mathrm{End}_{D_n}\left(V\left(2, \frac{n-1}{2}\right)^{\otimes k}\right)$ is INJECTIVE

2) $\ker \varphi = 0$

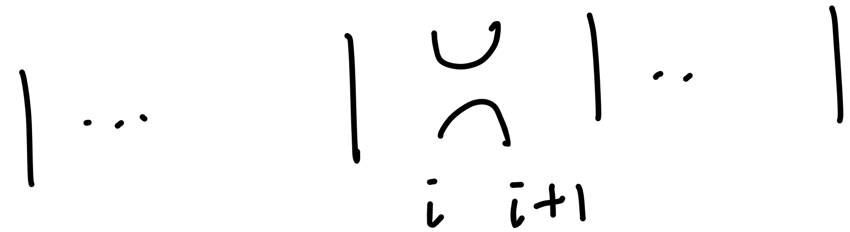
3) $x \in TL_k(q) \quad \varphi(x) = 0 \implies x = 0$

4) if $x \neq 0$ Then $\varphi(x) \neq 0$ i.e.

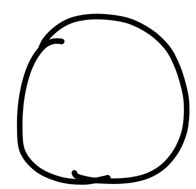
$$\boxed{\exists\ v \in V\left(2, \tfrac{n-1}{2}\right)^{\otimes k} \quad \text{s.t.} \quad x \cdot v \neq 0}$$

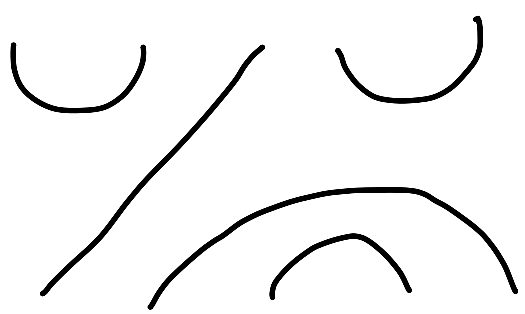
The action is FAITHFUL

a diagrammatic presentation

$t_i =$ 

product $\rightsquigarrow$ vertical stacking

up to
- a) homotopy equivalence
- b)  $= \delta$

e.g.  $\in D_n$

$\boxed{\text{“noncrossing matching”}}$

General idea:

D_i : Temperley-Lieb diagrams

Suppose $\sum \alpha_i D_i \neq 0$ choose D : HIGHEST with $\alpha \neq 0$

Claim

$$\boxed{\begin{array}{l} \exists \text{ suitable } v \in V(2,\tfrac{n-1}{2})^{\otimes k} \text{ s.t. } D.v \neq 0 \\ \text{AND other } D_i \text{ acts as zero} \end{array}}$$

Setup $\bar{t} \in \{1,2\}^k$ $\bar{t} = (i_1, i_2, \dots, i_k)$

$$v_{\bar{t}} = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}$$

Combinatorial Description

Prop. (BBKNZ)

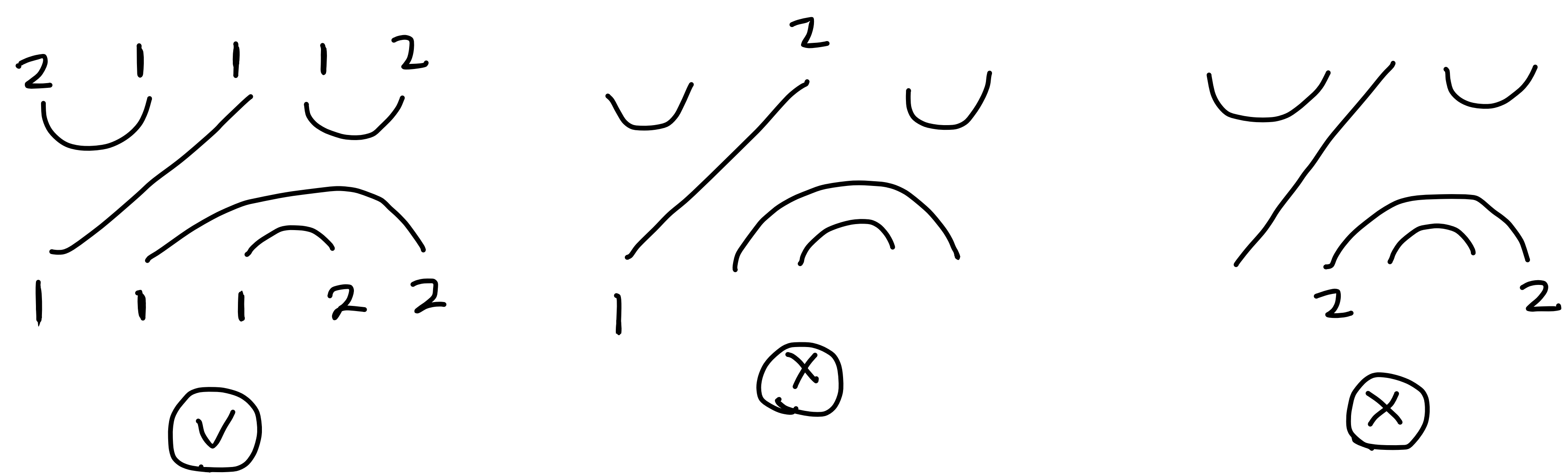
$i, j \in \{1,2\}^k$. D: TL-diagram

v_j appears in $D . v_i$ with NONzero coefficients

$\Longleftrightarrow$ $\overset{j}{\underset{i}{\boxed{D}}}$ is CONSISTENTLY LABELED.

(arcs: distinct integers
thru strands: same integers)

e.g.



Induction Step

Prop. (BBKNZ)

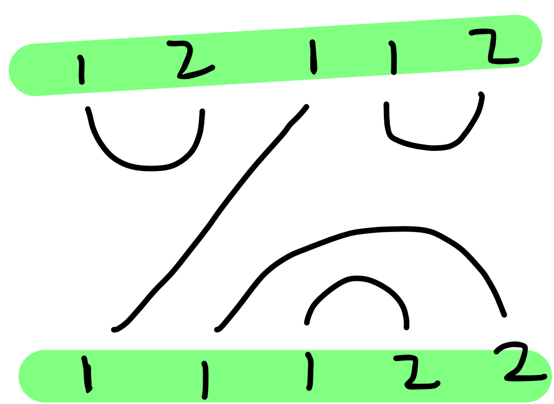
fix D : TL-diag

$\exists\ \underline{i}, \underline{j} \in \{1,2\}^k$ with

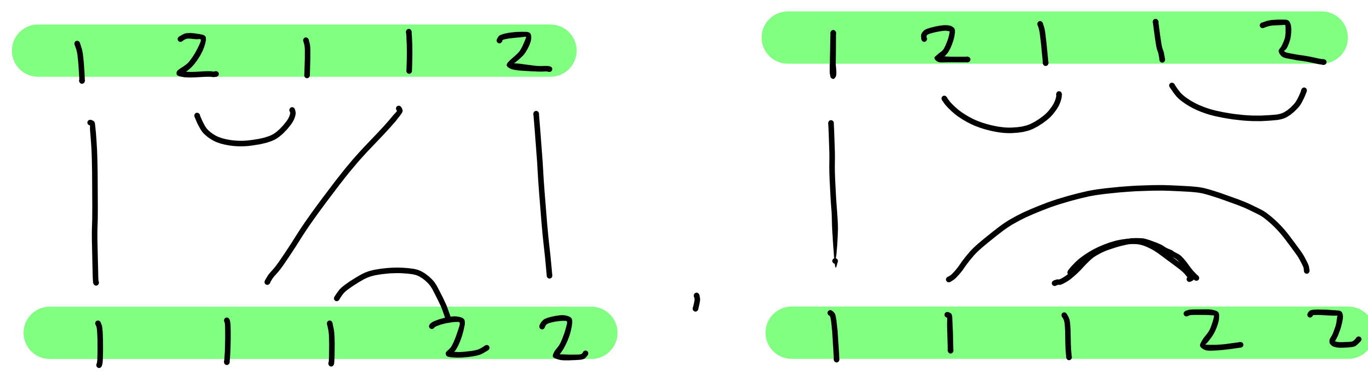
① $\underline{j}$ / $\boxed{D}$ / $\underline{i}$ consistent

② if $\underline{j}$ / $\boxed{E}$ / $\underline{i}$ is also consistent then $E \geq D$

e.g.



$\rightsquigarrow$



both diagrams are HIGHER

Surjectivity Statement

$V(2, \frac{n-1}{2})^{\otimes k}$: semisimple when $k \leq n-1$

Prop (BBKNZ)

$$\boxed{\begin{gathered} \dim TL_k(\xi) = \dim \mathrm{End}_{D_n}\left(V(2, \tfrac{n-1}{2})^{\otimes k} \right) \\ \text{when} \quad 1 \leq k \leq 2(n-1) \end{gathered}}$$

Note: easy when semisimple.

use $\quad TL_k(\xi) \xrightarrow{\sim} \mathrm{End}_{U(sl_2)}(V^{\otimes k})$ $\quad \underset{\uparrow}{V}$ natural $U_q(sl_2)$-mod