From Link Homology To Representations of The Symmetric Groups

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FULL DISCLAIMER

I'm a PURE Mathematician,

Intellectual persuits
unsolved conjectures

physics
industrial
Applications

e.g. Theory

Applications

Group Theory, Elliptic Curve

Cryptography

~ AHEAD OF ~>

Riemmanian Geometry

Einstein's Theory of Relativity

A bridge between two worlds

topology

link homology

conformal field theory

Topological Quantum Field Theory

Representation Theory

algebra

combinatorics

Quantum Groups

supersymmetry

" Pictures "

" matrices "

" Tableaux "

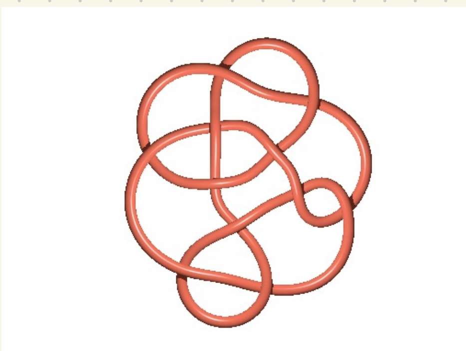
A Crash Course on Knot Theory

Q Compare the knots

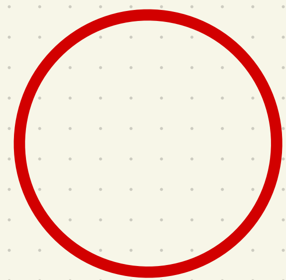
1.  $\rightsquigarrow$ 

Yes !

The Trefoil Knot

2. 

Conway Knot

$\rightsquigarrow$ 

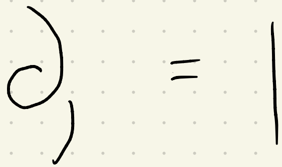
Unknot

Not in $\mathbb{R}^3$

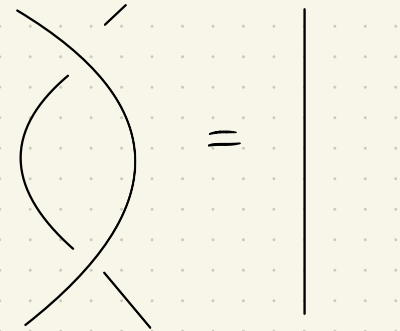
but in B^4 ?

Reidemeister Moves

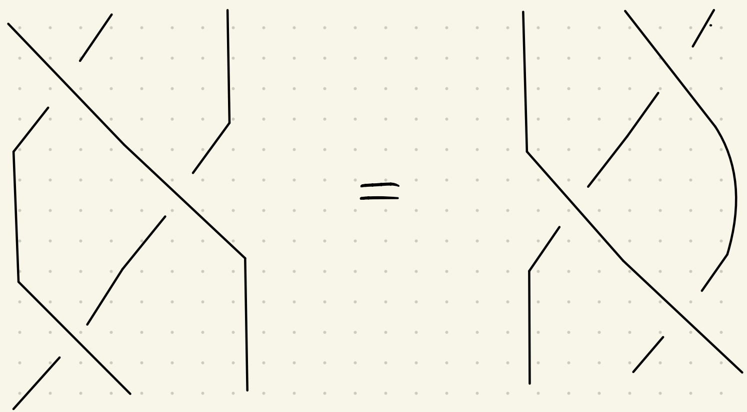
R1.



R2



R3



← (Yang-Baxter Relations)

Takeaway:

Knots are built from

CROSSINGS !

Full characterization of equivalent knots

R-Moves $\Longrightarrow$ SAME KNOTS !

Theorem (Reidemeister)

SAME KNOTS $\Longrightarrow$ existence of R-moves

i.e. No more complicated moves to relate two equivalent knots.

"divide & conqueur"

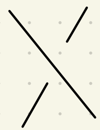
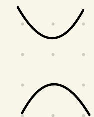
KnoT A $\Leftrightarrow$ KnoT 1 $\Leftrightarrow$ KnoT 2 $\Leftrightarrow$ ········· $\Leftrightarrow$ KnoT n $\Leftrightarrow$ KnoT B

have the same property

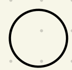
Knot Invariants

Jones Polynomials : link $\rightsquigarrow$ polynomial (Vaughan Jones)

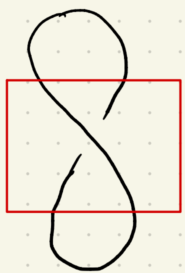
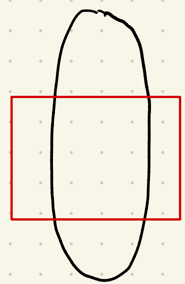
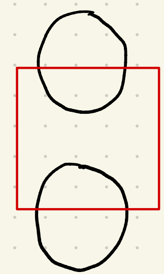
Algorithm :

 $=\ q$ $+\ q^{-1}$ 

Skein Relation

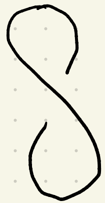
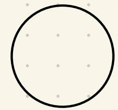
 $= -(q+q^{-1})$ (& + normalization)

e.g.

 $=\ q$  $+\ q^{-1}$ 

$= -q(q+q^{-1}) + q^{-1}(q+q^{-1})^2$ $\rightsquigarrow$ simplify !

Well-definedness

 =  = $-(q+q^{-1})$

"divide & conqueur"

Knot A $\Longleftrightarrow$ Knot 1 $\Longleftrightarrow$ Knot 2 $\Longleftrightarrow$ .... $\Longleftrightarrow$ Knot n $\Longleftrightarrow$ Knot B

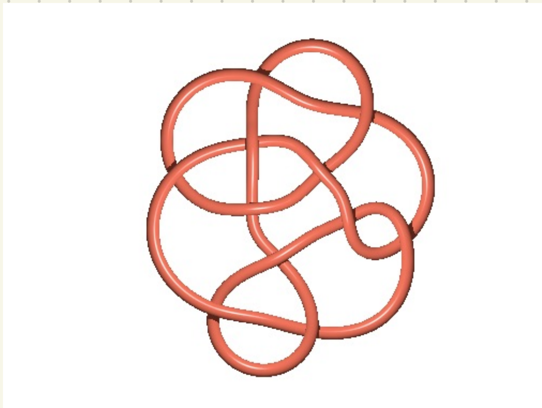
have the same Jones polynomial

Remark:

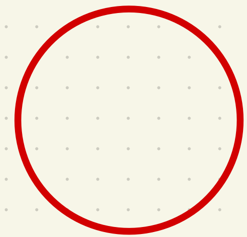
Knot A $\neq$ Knot B

$$\text{Knot A} \rightsquigarrow q+q^{-1} \qquad \text{Knot B} \rightsquigarrow q^{2}+1+q^{-2}$$

The Conway Slice Problem



$\neq$ in B^4



using Link invariants by Lisa Piccirillo

Representation Theory

The <u>Symmetric Group</u> : all permutations on $1, 2, \cdots, n$

e.g. $n = 4$. $\quad 1\ 2\ 3\ 4$, $\quad 1\ 2\ 3\ 4$, $\quad 1\ 2\ 3\ 4$, $\cdots$

24 permutations !

A <u>Representation</u> : to represent them as matrices,

e.g. $\quad 1\ 2\ 3\ 4 \qquad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

A matrix $\Longleftrightarrow$ A linear transformation on a vector space V

$$V = \mathbb{C}^4 = \mathbb{C}\text{-span}\langle \vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4 \rangle$$

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

as a linear trans.

$f(\vec{e}_1) = \vec{e}_4$

1 2 3 4

$$\vec{e}_1 \rightsquigarrow \vec{e}_4$$
$$\vec{e}_2 \rightsquigarrow \vec{e}_3$$
$$\vec{e}_3 \rightsquigarrow \vec{e}_2$$
$$\vec{e}_4 \rightsquigarrow \vec{e}_1$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$1 \quad 2 \quad 3 \quad 4$ $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$1 \ 2 \ 3 \ 4$ $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ $\rightsquigarrow$ ONE representation.

$\vdots$

A list of 24 matrices for $n=4$

Two Famous Bases $(n = 6)$

Tableaux Basis

indexing set :

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline\end{array}$$

$$\cancel{\begin{array}{cccc} 1 & 2 & 3 & 4 \\ A & B & C & D \end{array}}$$

e.g. Permutation $1\ 2\ 3\ 4\ 5\ 6$

out of $6! = 720$ permutations

$$S = \begin{bmatrix} -1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

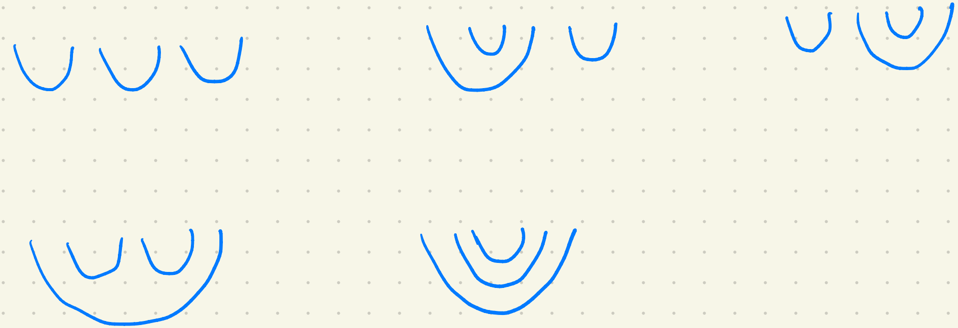
as a linear trans. $\rightsquigarrow$

$$f\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline\end{array}\right) = (-1)\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline\end{array} + 1 \cdot \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline\end{array}$$

Web Basis

(Temperley-Lieb)

diagrams



e.g.

Permutation 1 2 3 4 5 6

$$S' = \begin{bmatrix} -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Different Representations ?

Permutation $\overset{\curvearrowleft}{1\ 2}\ 3\ 4\ 5\ 6$

under tableaux basis $\leftsquigarrow$ $M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $\rightsquigarrow$ under web basis

$$S = \begin{bmatrix} -1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \longleftarrow\!\!\sim\!\sim\!\sim \quad S' = \begin{bmatrix} -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Conjugation by M

$S' = MSM^{-1}$

base change

Main Questions

Goal: Study M

e.g. Kazhdan - Lusztig conjecture (Base change for Hecke algebras)

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Properties :

1. Upper triangular (Russell - Tymoczko 2018)

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

② 1's on the diagonal (Russell - Tymoczko 2018)

Proven Conjectures:

③ Non-negative entries (Rhoades 2019)

④ Positive entries above diagonal (Im - Z. 2019)

Caveat: Partial Order

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

labeling row:

1	2	5
3	4	6

(2^{nd})

labeling column:

1	3	4
2	5	6

(3^{rd})

$2 < 3$, but

1	2	5
3	4	6

and

1	3	4
2	5	6

are NOT comparible!

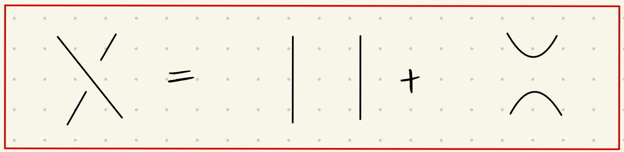
Zero is NOT an uppertriangular entry

Key Idea

Algorithm of computing M

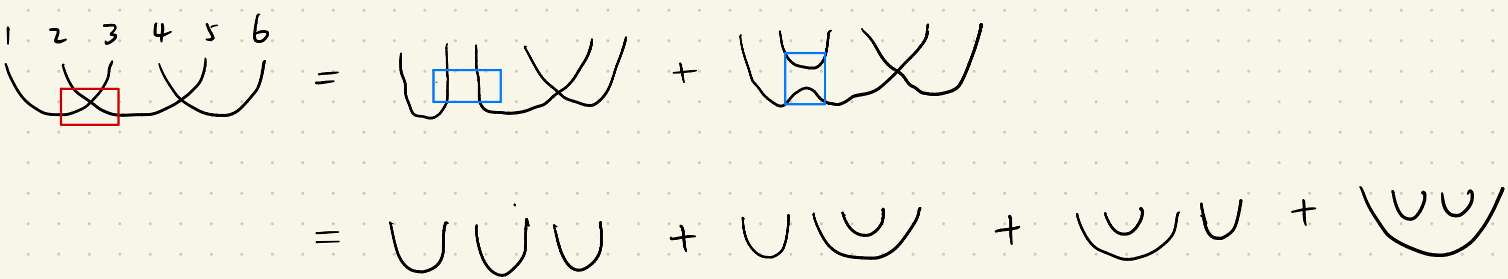
$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Skein Relation



Quantum variable

$q \rightsquigarrow 1$

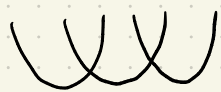


Key Breakthroughs

- Coefficient (want to compute)

  ||

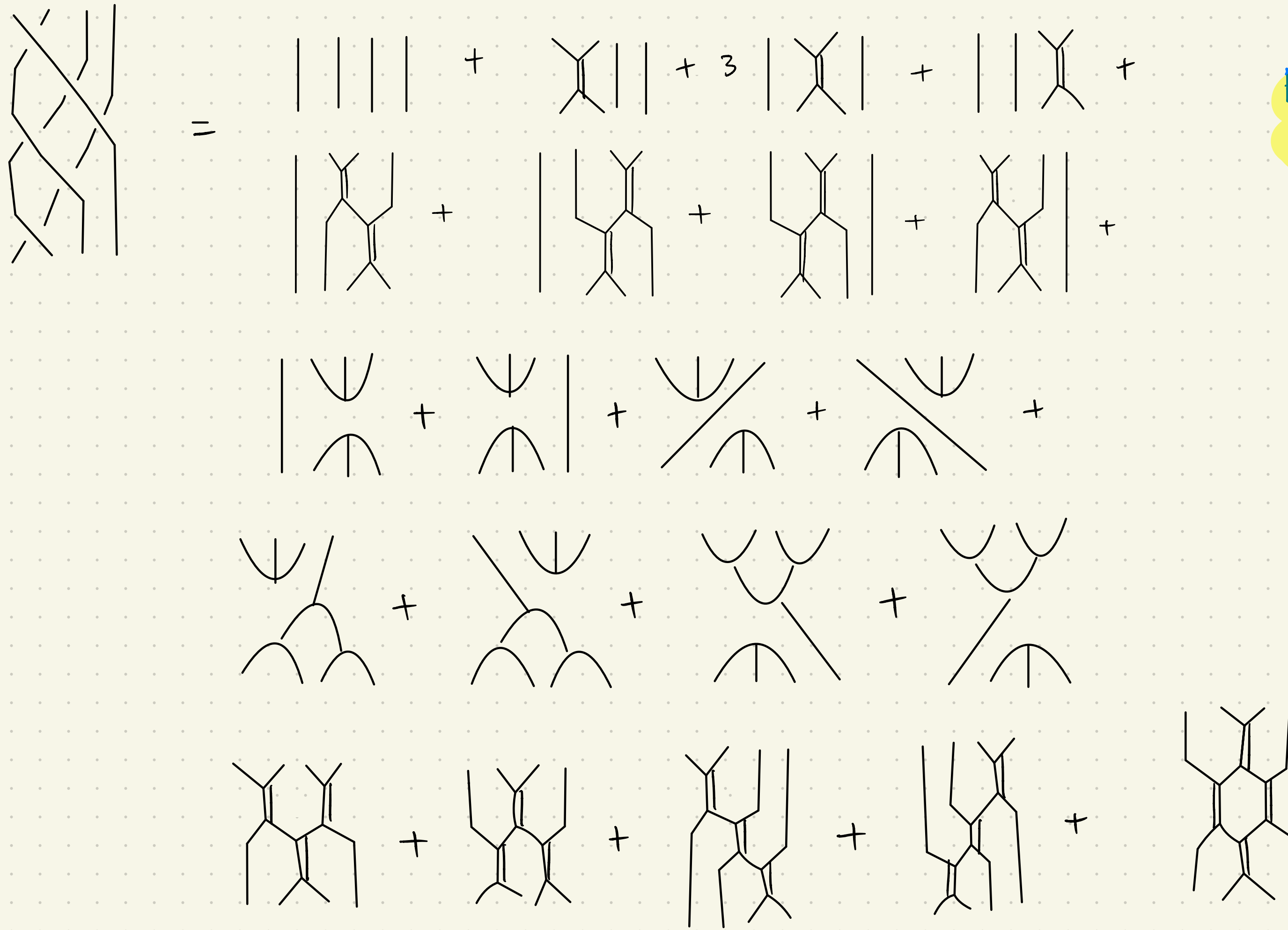
  # of ways (counting problem) nonnegative by default.

from  to 

- e.g. n = # of particals in the universe

existence of one way = positive count

Future Work



THANK YOU !