Soergel Bimodules

Perverse Sheaves

Stratified Space / flag varieties

A topological space X has a **stratification** if

- $\exists$ finite poset Λ
- $X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$, X_λ locally closed set (open $\cap$ closed)
- each X_λ is a smooth open manifold (with complex dimension)
- $\overline{X_\mu} = \bigsqcup_{\lambda \leq \mu} X_\lambda$
- each X_λ is called a **Stratum**

e.g. Flag Varieties

G: reductive group T: max Torus

$W = N_G(T)/T$ Weyl gp. B: Borel

G/B: flag variety

special e.g. In Type A. $GL_n(\mathbb{C})$

a **flag** is $V^\bullet = (V_0, V_1, \dots, V_n)$

s.t. $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n$, $\dim V_i = i$

special e.g. $n=3$

$$\begin{bmatrix} 1 & 3 & 7 \\ 2 & 0 & 6 \\ 0 & 4 & 5 \end{bmatrix} = [\ \vec{v}_1\ \ \vec{v}_2\ \ \vec{v}_3\ ]$$

$V_0 = 0,\quad V_1 = \langle \vec{v}_1 \rangle,\quad V_2 = \langle \vec{v}_1, \vec{v}_2 \rangle,\quad V_3 = \langle \vec{v}_1, \vec{v}_2, \vec{v}_3 \rangle = \mathbb{C}^3$

$V^\bullet = (V_0, V_1, V_2, V_3)$

Claim right multiplication by B preserves the flag

e.g.

$$[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]\begin{bmatrix} 3 & 4 & 7 \\ & 1 & 8 \\ & & 2 \end{bmatrix}$$

$$= [\,3\vec{v}_1 \quad 4\vec{v}_1 + \vec{v}_2 \quad 7\vec{v}_1 + 8\vec{v}_2 + 2\vec{v}_3\,]$$

represents the same flag $V^\bullet$

Theorem (Bruhat Decomposition)

$$G/B = \bigsqcup_{w \in W} B\dot{w}B/B$$

is a stratification of G/B

with
- $\le$ Bruhat order
- $\overline{B\dot{w}B/B} = \bigsqcup_{x \le w} B\dot{x}B/B$

and
- $\dim B\dot{x}B/B = \ell(x)$
- each $B\dot{w}B/B$ is called a Schubert Cell

e.g. Type A $\qquad G = \bigsqcup_{w \in W} B\dot{w}B$

left mult by B: add (a scalar multiple of) a row to a row above

right mult by B: add (a scalar multiple of) a column to a column to the right

any matrix $\xrightarrow{\text{"Gaussian Elimination"}}$ permutation matrix

- The standard flag is rep. by $\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$

i.e. $\quad \mathrm{std}^\bullet = (\mathrm{std}^i) \qquad \mathrm{std}^i = \langle e_1, \dots, e_i \rangle$

- The position of a flag $V^\bullet$ is a matrix (f_{ij})

$$f_{ij} = \dim (V^i \cap \mathrm{std}^j)$$

Claim
- The position characterizes the Schubert cells
  (i.e, $V^\bullet, W^\bullet$ in the same cell $\Longleftrightarrow$ $f_{ij} = f_{ij}' \ \forall i,j$)
- f_{ij} increases from a cell to its closure
  (i.e. from $B\dot{w}B/B$ to $B\dot{x}B/B$ with $x \leq w$)

e.g. $GL_3(\mathbb{C}) \qquad s = (12) \quad t = (23)$
- (f_{ij}) remains constant within each cell

BsB
$$\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} B \quad \text{or} \quad \begin{bmatrix} 4 & 1 & 7 \\ 1 & & 6 \\ & & 1 \end{bmatrix} B$$
$$\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \qquad\qquad \vec{w}_1 \ \vec{w}_2 \ \vec{w}_3$$

$V^1 \cap \mathrm{std}^1 = 0$
$V^2 \cap \mathrm{std}^1 = \langle \vec{e}_1 \rangle$
$V^3 \cap \mathrm{std}^1 = \langle \vec{e}_1 \rangle$
$V^1 \cap \mathrm{std}^2 = \langle \vec{v}_1 \rangle$
$V^2 \cap \mathrm{std}^2 = \langle \vec{e}_1, \vec{e}_2 \rangle$
$V^3 \cap \mathrm{std}^2 = \langle \vec{e}_1, \vec{e}_2 \rangle$

$$f_{ij} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$W^2 \cap \mathrm{std}^1 = \langle \vec{w}_1 \rangle$
$W^2 \cap \mathrm{std}^2 = \langle \vec{e}_1, \vec{e}_2 \rangle$
$W^2 \cap \mathrm{std}^3 = \mathbb{C}^3$

- (f_{ij}) in all cells

	w	(f_{ij})	
$B\,id\,B$	$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$	
BsB	$\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$	
BtB	$\begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$	$V_2 \cap std^2 = \langle e_1 \rangle$
$BstB$	$\begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$	$V_2 \cap std^1 = 0$
$BtsB$	$\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$	$V_1 \cap std^2 = 0$
$BstsB$	$\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$	

- f_{ij} increases along cells in the closure

  BwB/B locally closed set

e.g. $f_{22} = 2 \iff \dim V_2 \cap \mathrm{std}^2 = 2$

$\iff \dim(V_2 + \mathrm{std}^2) = \dim V_2 + \dim \mathrm{std}^2 - \dim V_2 \cap \mathrm{std}^2$
$= 2$

$$\begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{e}_1 & \vec{e}_2 \end{array}$$
$$\begin{bmatrix} x_{11} & x_{12} & 1 & 0 \\ x_{21} & x_{22} & 0 & 1 \\ x_{31} & x_{32} & 0 & 0 \end{bmatrix}$$ has rank 2

$\iff$ · any 3-minors $= 0$ closed condition

· $\exists$ 2-minors $\neq 0$ open condition

closure

$f_{22} = 1 \iff$

$\dim(V_2 + \mathrm{std}^2) = 3$ · $\exists$ 3-minors $\neq 0$

- Partial flags

given $\underline{d} = (d_1, \dots, d_s)$ $\Sigma d_i = N$ s.t. $\dim V_i = \sum_{j=1}^{i} d_j$

A partial flag with dimension vector d is $V^\bullet = (V_0, V_1, \dots, V_s)$

G/P with $P = \begin{bmatrix} GL_{d_1} & * & * & \cdots & * \\ & GL_{d_2} & * & \cdots & \vdots \\ & & \square & & * \\ & & & \ddots & \\ & & & & GL_{d_s} \end{bmatrix}$

standard flag $\mathrm{std}_{\underline{d}}$: rep by id matrix e.g. $Gr(2,4) = \{\text{partial flags of dim } (2,2)\}$

Fact stratification of $X = Gr(2,4)$

$$X = X_0 \sqcup X_1 \sqcup X_2 \qquad V = \langle e_1, e_2 \rangle$$

$$X_i = \{ W \in X \mid \dim W \cap V = i \}$$

closure structure : $\overline{X_i} = \sum_{j \geq i} X_j$

i.e. $\overline{X_2} = X_2$

$\overline{X_0} = X$

Resolution of singularities

Resolution of Singularities

A proper map of stratified space $f: Y \to X$ is such that

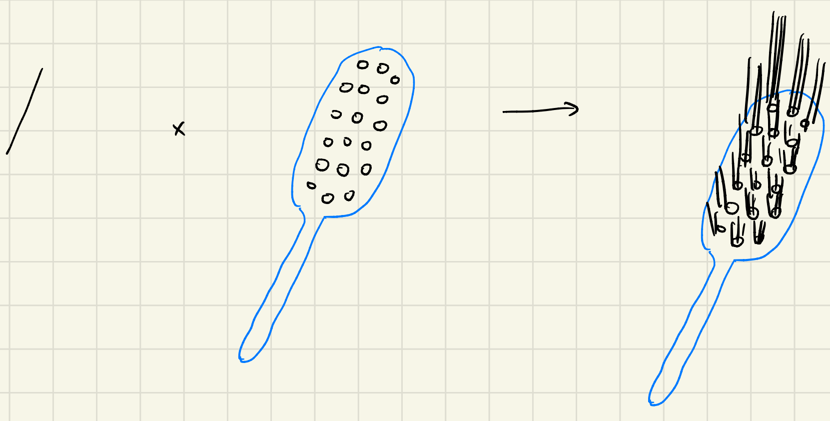
Each restriction $f^{-1}(X_\lambda) \longrightarrow X_\lambda$

is a topologically local trivial fiber bundle with compact fiber F_λ

A trivial fiber bundle is a projection

$$Z \times X \longrightarrow X$$

with Z being the fiber



f is called a resolution of singularities if

$f^{-1}(X_\lambda) \xrightarrow{\sim} X_\lambda$ on each open stratum

i.e. the fiber is trivial (i.e a point)

Induced set and Bott-Samuelson varieties

Let G gp X topological space $G \curvearrowright X$

$H \trianglelefteq G$. define the induced set

$$G \overset{H}{\times} X := G \times X / \sim$$

$$\sim : \quad (gh, x) \sim (g, hx)$$

Fact: if $H \curvearrowright X$ simply transitively

$$G \overset{H}{\times} X \longrightarrow G/H \quad \text{with fiber } X$$

$$(g, x) \longmapsto gH$$

preimage of H: $H.X = X$

Let $P_s = (B, \dot{s})$

e.g. in GL_n

$$P_s = \begin{array}{c} \qquad\qquad s \\ \begin{bmatrix} * & & & & & \\ & \ddots & & & * & \\ & & * & & & \\ & & & \begin{bmatrix} * & * \\ * & * \end{bmatrix} & & \\ & & & & * & \\ & & & & & \ddots & \\ & & & & & & * \end{bmatrix} \end{array}$$

For $\underline{w} = (s_1, \dots, s_t)$ a reduced expression

The Bott-Samuelson variety

$$Y(\underline{w}) = P_{s_1} \overset{B}{\times} P_{s_2} \overset{B}{\times} \cdots \overset{B}{\times} P_{s_t}/B$$

$P_{s_{t-1}} \curvearrowright P_{s_t}/B$ $\qquad$ $P_{s_{t-1}} \overset{B}{\times} P_{s_t}/B = \{(g_1, g_0B)\}/\sim$

$P_{s_{t-2}} \curvearrowright P_{s_{t-1}} \overset{B}{\times} P_{s_t}/B$ $\quad \cdots\cdots$

$$P_{s_{t-2}} \overset{B}{\times} P_{s_{t-1}} \overset{B}{\times} P_{s_t}/B = \{(g_2, g_1, g_0B)\}/\sim$$

Fact $\quad Y(\underline{w})$ iterated $\mathbb{P}^1_{\mathbb{C}}$-bundle over a point

$$P_{s_i} \overset{B}{\times} X \longrightarrow P_{s_i}/B \quad \text{with fiber } X$$

but $\quad P_{s_i}/B \simeq Gl_2/B = \{\text{affine line in } \mathbb{C}^2\}$
$= \mathbb{P}^1_{\mathbb{C}}$

Hence $\quad Y(\underline{w}) = \mathbb{P}^1_{\mathbb{C}} \times (\mathbb{P}^1_{\mathbb{C}} \times (\mathbb{P}^1_{\mathbb{C}} \times \cdots \quad)$

is smooth

- mult: $\quad Y(\underline{w}) \longrightarrow G/B$

$$(g_1, g_2, \cdots, g_tB) \longmapsto g_1 \cdots g_tB$$

Claim image is $\overline{B\dot{w}B/B}$

and $Y(\underline{w}) \longrightarrow \overline{B\dot{w}B/B}$

is a resolution of singularities

Constructible Sheaves

A sheaf on a topological space X is a functor

$$\mathcal{F}: \mathrm{Top}(X) \longrightarrow \text{Vector Spaces}/\mathbb{C}$$

morphisms: reverse inclusion

$U \subseteq V \qquad V \to U$

$x \in X$ the Stalks at x is

$\mathcal{F}_x$ = direct limit of $\mathcal{F}(U)$ for open nbhd U of x

a sheaf $\mathcal{F}$ on a stratified space X is Constructible if

The Stalks $\mathcal{F}_x$ are constant on each stratum, and

	$\mathcal{F}_x$
λ	$\mathbb{C}^{g_\lambda}$
μ	$\mathbb{C}^{g_\mu}$
ν	$\mathbb{C}^{g_\nu}$

$D(X)$: The derived category of constructible sheaves on X

	-2	-1	0	1	2
λ	$*$	$*$			
μ	$*$				
ν					

e.g. The constant sheaf $\mathbb{C}_X \in D(X)$, viewed as being concentrated in homological deg 0

	$\ldots, -2, -1$	0	$1, 2, \ldots$
λ	0	$\mathbb{C}$	0
μ	0	$\mathbb{C}$	0
ν	0	$\mathbb{C}$	0

Let $f: Y \to X$ a continuous map

$\mathcal{F}$: sheaf on Y

$$\mathrm{Top}(X) \xrightarrow{f^{-1}} \mathrm{Top}(Y) \xrightarrow{\mathcal{F}} \text{Vector Spaces}$$

Composition is the push forward of $\mathcal{F}$ via f,

denoted as $f_*\mathcal{F}$, is a sheaf on X

similarly $f_*: D(Y) \to D(X)$

Theorem (Push forward of the constant sheaf)

$f: Y \to X$ proper map of stratified spaces

then $f_* \mathbb{C}_Y$ has the table

	$-2, -1, 0, 1, 2, \dots$
λ	$H^*(F_\lambda)$
μ	$H^*(F_\mu)$
ν	$H^*(F_\nu)$

Example

recall : mult : $Y(s,t,s) \longrightarrow \overline{BstsB/B}$

$$\overline{BstsB/B} = Gl_3/B$$

Compute $\text{mult}_* \mathbb{C}_{Y(s,t,s)}$

First find fibers of mult on each stratum

$$P_s \overset{B}{\times} P_t \overset{B}{\times} P_s/B \longrightarrow G/B$$

$$P_s = BsB \sqcup BidB$$

non-zero

$$\begin{bmatrix} * & * & \\ * & * & \\ & & * \end{bmatrix} \sqcup \begin{bmatrix} * & * & \\ 0 & * & \\ & & * \end{bmatrix}$$

$\mathbb{P}^1_{\mathbb{C}} \setminus \{pt\}$ std for $\mathbb{C}^2$

$$\text{“}\begin{bmatrix} 1 & & \\ a & 1 & \\ & & 1 \end{bmatrix}\text{”} \quad \text{“}\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}\text{”}$$

$$\begin{bmatrix} 1 & & \\ a & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & \\ & b & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ c & 1 & \\ & & 1 \end{bmatrix}$$

$$abc \neq 0 \rightsquigarrow \begin{bmatrix} 1 & & \\ a+c & 1 & \\ bc & b & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \rightsquigarrow BstsB$$

$\mathbb{P}^3_{\mathbb{C}} \longrightarrow \mathbb{P}^3_{\mathbb{C}}$ fiber is $\{pt\}$

same for st, ts, t

interesting cases: $BidB$ and BsB

$$\begin{bmatrix} 1 & & \\ a & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ -a & 1 & \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \longrightarrow BidB$$

$$\mathbb{P}^1_{\mathbb{C}} \longrightarrow pt$$

fiber: $\mathbb{P}^1_{\mathbb{C}}$

also

$$\begin{bmatrix} 1 & & \\ a & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ b & 1 & \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} & 1 & \\ a+b & 1 & \\ & & 1 \end{bmatrix} \underset{\text{if } a \neq -b}{\rightsquigarrow} \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} \rightsquigarrow BsB$$

$$\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$$

fiber : $\mathbb{P}^1_{\mathbb{C}}$

Hence $\mathrm{mult}_* \mathbb{C}_{Y(s,t,s)}[3]$ (as a sheaf on G/B)

F_λ	λ	-3	-2	-1	0
$\{pt\}$	sts	$\mathbb{C}$	0	0	0
$\{pt\}$	st	$\mathbb{C}$	0	0	0
$\{pt\}$	ts	$\mathbb{C}$	0	0	0
$\{pt\}$	t	$\mathbb{C}$	0	0	0
$\mathbb{P}^1_{\mathbb{C}}$	s	$\mathbb{C}$	0	$\mathbb{C}$	0
$\mathbb{P}^1_{\mathbb{C}}$	id	$\mathbb{C}$	0	$\mathbb{C}$	0

$\swarrow H^*(F_\lambda)$

Perverse Sheaves

A sheaf $\mathcal{F}$ is perverse if

F_λ	λ	-3	-2	-1	0
$\{pt\}$	sts	$\mathbb{C}$	0	0	0
$\{pt\}$	st	$\mathbb{C}$	0	0	0
$\{pt\}$	ts	$\mathbb{C}$	0	0	0
$\{pt\}$	t	$\mathbb{C}$	0	0	0
$\mathbb{P}^1_{\mathbb{C}}$	s	$\mathbb{C}$	0	$\mathbb{C}$	0
$\mathbb{P}^1_{\mathbb{C}}$	id	$\mathbb{C}$	0	$\mathbb{C}$	0

- supported on some twisted diagonal
  $(\lambda, -d_\lambda)$
- Same pattern for the Verlinde dual of $\mathcal{F}$

"Intersection Cohomology Sheaf" if further:

"pivots" are $\mathbb{C}, 0, 0, 0, \ldots$

F_λ	λ	-3	-2	-1	0
$\{pt\}$	sts	$\mathbb{C}$	0	0	0
$\{pt\}$	st	$\mathbb{C}$	0	0	0
$\{pt\}$	ts	$\mathbb{C}$	0	0	0
$\{pt\}$	t	$\mathbb{C}$	0	0	0
$\mathbb{P}^1_{\mathbb{C}}$	s	$\mathbb{C}$	0	$\mathbb{C}$	0
$\mathbb{P}^1_{\mathbb{C}}$	id	$\mathbb{C}$	0	$\mathbb{C}$	0

∴ Not an IC sheaf

Example

find a resolution of singularity for $Gr(2,4)$ and compute the pushforward of the constant sheaf