In the paper by Lai et al., the authors conducted a study to explore the question of what aspects would contribute to mathematicians' perception of a good pedagogical proof. They first stressed the importance of pedagogical proofs by referencing the usual setup of an advanced mathematics course, and the communication purposes of proofs in general. Their study was carried out in two parts. In the first part, they tried to explore possible factors by interviewing 8 professors from the top 25 mathematics graduate programs. They showed them two proofs with intentional errors and other pitfalls and asked them to revise them for pedagogical purposes. They then categorized and examined the frequency of each type of revision, from which they generated four hypotheses about factors contributing to the effectiveness of a pedagogical proof. Based on such results, they carried out the second study, in which they suggested five revised versions of a proof associated with each hypothesis, and asked 110 mathematicians through the internet about the effectiveness of each. After analyzing the data, they concluded that hypothesis 1,2 and 4 were accepted, that both having introducing and concluding sentences on the logical structure of the proof and better formatting would help the effectiveness, while having redundancy will decrease the effectiveness for pedagogical purposes. They further mentioned possible points for future study, where one could focus on the evidence-based study in student abilities.

In the paper by Seldon and Seldon, the authors proposed three factors contributing to students' ability of constructing a proof that are the most difficulty to teach. They begun their discussion with a theoretical assumption that there are two components of a proof: the formal-rhetorical component and the problem-centered component. They associated these two factors to the two types of cognition in dual-process theory, one that is slow, evolutionary recent and more conscious, the other fast, evolutionary ancient and more intuitive. They then proceeded to the discussion of the three factors. The first factor is the behavioral schemas that students develop during practice, by which they could use results without revoking the definition explicitly. They then discussed how self-efficacy can help strengthen students' ability to construct a proof, in ways including motivating students to work on more challenging problems and cultivating a deeper interest and commitment. When discussion the self-efficacy factor, they mentioned its interplay with the third persistence factor: that self-efficacy will make students less likely to get frustrated with set-backs. To illustrate the last two factors, they referenced the study by Dr. Savic on nine mathematicians, in examination of their approach to a hypothetical proof. By citing the data of one professor, they concluded mathematicians in general are more willing to try new ideas that don't lead to a useful result in an obvious way, able to walk away from a problem temporarily, and are more persistent as a result of higher self-efficacy.

They further gave implications on teaching based on such three factors, and recommended a teaching style similar to Moore's that introduces proofs at an early stage, and also promoted a proof framework that separates the problem-centered part from the formal rhetorical part.

First of all, I am impressed by the choice of subject in Lai's paper, since proofs have been ignored in almost all of the papers we read. Second, even though all the factors in their paper seem intuitive to me, I appreciate they have a quantitative measure of the effectiveness, and their methodology of why "these should be the factors we are testing." I agree in totality with them that proofs should be essential in mathematical studies, or even in lower level mathematical studies, going back to our Monday discussion about K12/common core education. For example, I TAed for a professor that I personally respected very much, and to my surprise, his calculus class consisted of a large amount of proofs, and the students are able to receive it really well, even on tests. This is what inspired me to interpret David's analogy one day, that sometimes the teacher is showing the students how to make a clock(proving a theorem,) and the students would have a better time telling time(do problems.) I really liked the class and abandoned my old belief that proofs are "too much" for non-mathematics major students. This is in agreement with Seldon and Seldon's last section on the teaching implication as well. From the discussion in class, we talked about the logic table with ponies that doesn't involve the explicit mentioning of logic terms such as contrapositives. If I was a K12 teacher, I would try incorporating logic puzzles in my curriculum to develop such ability. Also, even at an early stage when students learn how to add fractions or use the distribution law, the laws could be introduced as theorems for students to prove for themselves instead of being handed down without justification. I have seen more mistakes on those basic properties in my teaching than I can be patient with, and I think its root lies in the lack of cultivation in their proof construction. Their lack of self-efficacy in proving those laws leads to their fear of mathematical reasoning, which in my opinion inhibits further learning.