In the paper by Tall, the author incorporated different theories and proposed three mathematical worlds to explain humans' transition to formal understanding of mathematics. He begun the discussion by introducing the properties in human minds that are genetically set before the humans were born: recognition, repetitions, and language, and argued they are crucial to form understanding. He then gave examples of experiences that humans have met before encountering a certain situation and how they could influence understanding. Based on such discussion, he proposed the three worlds in mathematics: the conceptual-embodied world that is perceived by the human and reflected in the human mind; the proceptual-symbolic world that is the compression of conceptual procedures in language or symbols; the axiomatic-formal world that is the logical and set-theoretical construction of the experiences. He related the symbolic world to the APOS theory and explained there is an embodied counterpart of such theory, where the process is the reflection of the effect of an imaginary object. Based on the three worlds, he proposed that teaching should follow the transition pattern of human understanding, which is built on the embodied world and transitioned to symbolic then formal world, and from formal back to embodied and symbolic. After discussing how the blending of different concepts may lead to understanding in both cases of the real numbers and calculus, he gave the course description of such a teaching example in calculus, in which concepts were built from the embodied world looking up, in which the property of differentiability is introduced as being local straight lines illustrated by computer graphing programs. He then discussed the association with proof construction research by scholars such as Webber and Pinto, where three kinds of student responses are present in correspondence with the three worlds. To complete the circle, he discussed situations scientists may go back to the embodiment and symbols after they understand the formal world.

In the paper by Oehtman at al., the authors first stressed the importance of understanding the concept of functions for students based on its wide usage in science, engineering and mathematics. They then pointed out several confusions students may have when learning functions, including viewing function as a procedure of plugging in numbers into a single formula, not being able to distinguish between functions and equations, or viewing functions as the physical shape of a real life object. They then referenced previous work of two authors of the paper and stated that students should achieve the understanding of functions as mappings between two sets, as well as the changing of the dependent variable and its rate of change according to the independent variable. To further illustrate the two goals, he referenced scholars Dubinsky and Harel and the APOS theory to categorize common understanding as Action based or Procedure based, and gave examples of student responses to characterize behaviors in each category. Based on these findings, they listed several recommendations for teaching, including asking questions that foster a procedural understanding, asking questions about a whole interval, or asking questions that promote a sense of covariance. To explain the last recommendation, they proposed a theoretical framework for the sense of covariance consisting of five levels(AM1-AM5) in a progressive manner, gave examples of student responses associated with each level, and suggested questions for teaching accordingly.

I find Tall's paper strongly related to the APOS theory and the concept image/concept definition paper, as pointed out by the author. I could see how they all come from the belief that mathematics is based on experiential world instead of axiomatic abstractions, and we should teach the students why mathematicians chose to abstract certain embodiment in a certain way. If the students are not motivated by any embodiment of the physical world, he may never see the point of formal definitions. In calculus classes, I find students are much better at understanding the definition of derivatives, after they were prompted to recall the slope of secant lines and interpret derivatives as the slope of the tangent lines. In agreement with Oehrman at al., when I was teaching precalculus, I find that students have a hard time understanding inverse functions, and they are usually satisfied with remembering the procedure of finding inverse function listed in their paper. Similar things also happen when they try to remember the rules of shifting graphs of function translations. I struggled with making them understand functions as a map, but since they lack the axiomatic set knowledge, I found it hard to get the idea across even when I drew the domain and codomain in circles to visualize them as packages. I will consider using some of the questions designed by the paper if I were to teach the same course again.