# Bases of The Specht Modules

( jt w/ M. S. Im )

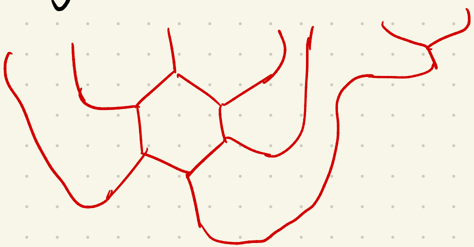
Reed College

April 2021

# Linear Algebra

More generally

→



→

our project

$v_1, v_2, v_3, \dots$

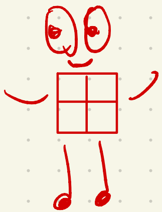
?

| | | |
|---|---|---|
| 1 | 2 | 6 |
| 3 | 4 | 7 |
| 5 | 8 | 9 |

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 5 | 6 | 7 |
| 3 | 4 | 8 | 9 | 10 |

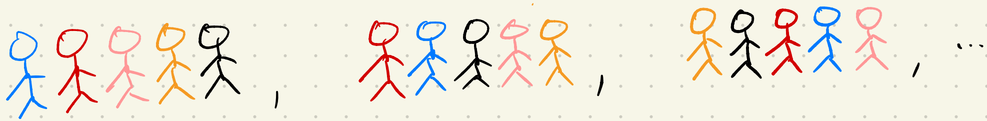
$w_1, w_2, w_3, \dots$

# I. A Crash course on Representation Theory



# The Symmetric Group

**Q** How many different arrangements ?

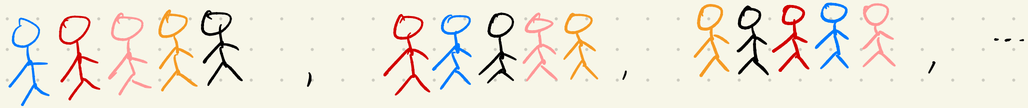


**A** $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$ ways !

**Def** The Symmetric Group $S_d$ consists of all Permutations on a $d$-element set.

# Book-Keeping

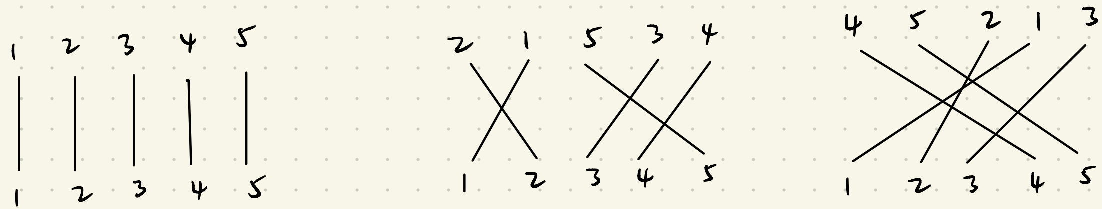
## Option 1



## Option 2

12345, 21534, 45213

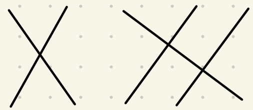
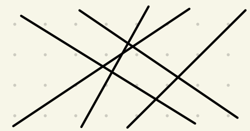
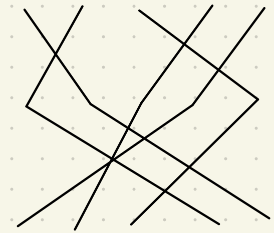
## Option 3

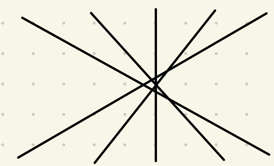



or simply,

**Def** Composition in $S_d$ is given diagrammatically

e.g.

 $\circ$  $=$  $g \circ f$

$=$ 

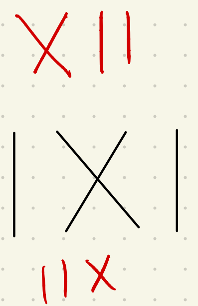
**Def** A group is a set with a composition.

# A Representation.

(i.e. Module for $S_n$)

permutation $\longmapsto$ matrix / linear map

e.g.



"adjacent swaps"

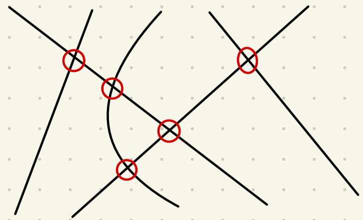
$$\longmapsto \quad \begin{array}{c} f \\ \shortparallel \end{array} \begin{bmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{bmatrix} = \begin{array}{cccc} f(\vec{e_1}) & f(\vec{e_2}) & f(\vec{e_3}) & f(\vec{e_4}) \\ \end{array} \left[ \vec{e_1} \ \ \vec{e_3} \ \ \vec{e_2} \ \ \vec{e_4} \right]$$

adjacent

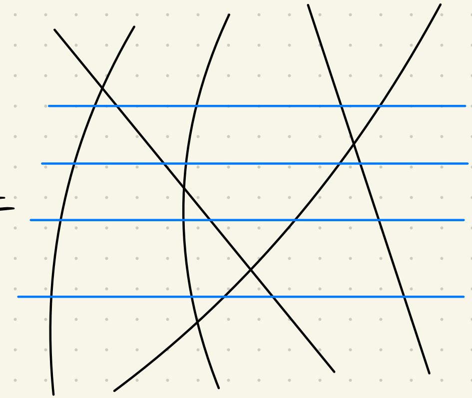
elementary row operation
of swapping rows 2 & 3

Idea: "simple transpositions" are building blocks of $S_n$

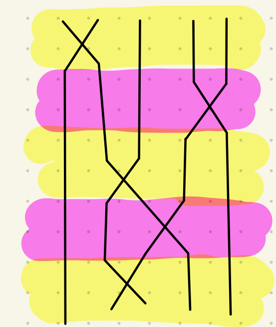
any permutation $= \prod$ simple transpositions e.g.



$=$

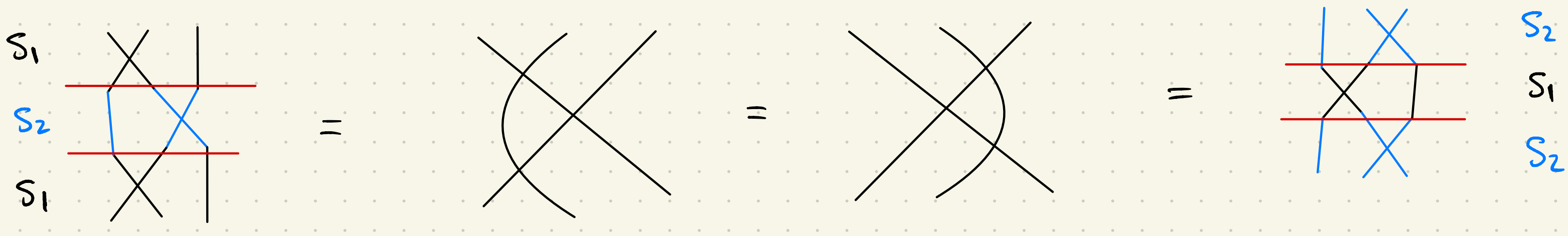


$=$



$$\mapsto \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & & & & \\ & 0 & 1 & & \\ & 1 & 0 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & & & & \\ & 0 & 1 & & \\ & 1 & 0 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

# Ambiguity / well-definedness




(Braid Relation)

It's okay ....

$$\begin{bmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{bmatrix}$$

$$s_1 s_2 s_1 = s_2 s_1 s_2$$

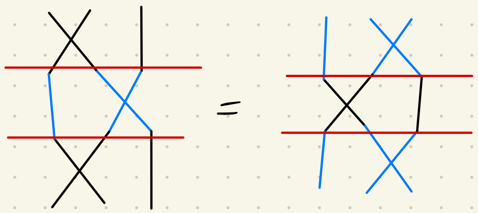
Nontrivial !

# Two (easy) Representations (1 dim'l)

e.g. ① any simple transposition $\mapsto [1]$

the Braid relations?

$$[1]\cdot[1]\cdot[1] = [1]\cdot[1]\cdot[1]$$



e.g. ② any simple transposition $\mapsto [-1]$

$$\text{permutation} \mapsto \begin{cases} [1] & \text{\# of crossings is even} \\ [-1] & \text{\# of crossings is odd} \end{cases}$$

the Braid relations?

$$[-1]\cdot[-1]\cdot[-1] = [-1]\cdot[-1]\cdot[-1]$$

# One Nontrivial Representation

( for $S_4$ )

Use $2 \times 2$ matrices ( 2-dim'l representation )

$$\begin{array}{c|cc} & v_1 & v_2 \\ \hline v_1 & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{bmatrix} \phantom{0} & \phantom{0} \\ \phantom{0} & \phantom{0} \end{bmatrix}$}} \\ v_2 & & \end{array}$$

new labels $\rightsquigarrow$

$$\begin{array}{c} \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \\ \\ \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \end{array} \begin{array}{cc} \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \\ \multicolumn{2}{c}{\begin{bmatrix} \phantom{0} & \phantom{0} \\ \phantom{0} & \phantom{0} \end{bmatrix}} \end{array} \qquad \boxed{\begin{array}{c|c} 2 & 1 \\ \hline 3 & 4 \end{array}}$$

STANDARD tableaux

entries increase $\rightarrow$ $\downarrow$

 $\mapsto \begin{bmatrix} -1 & -1 \\ & 1 \end{bmatrix}$

 $\mapsto \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$

 $\mapsto \begin{bmatrix} -1 & -1 \\ & 1 \end{bmatrix}$

braid relations

$$\begin{bmatrix} -1 & -1 \\ & 1 \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\begin{bmatrix} -1 & -1 \\ & 1 \end{bmatrix} = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\begin{bmatrix} -1 & -1 \\ & 1 \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$$

$$\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\begin{bmatrix} -1 & -1 \\ & 1 \end{bmatrix}\begin{bmatrix} & -1 \\ 1 & \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ & 1 \end{bmatrix}\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$$

# Recipe

$$XII \curvearrowright \quad \operatorname{span}\left\{ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \right\} \quad \text{via} \quad ?$$

$$f(\vec{v}_2) = f\left( \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \right) = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & 4 \\ \hline \end{array}$$

$$= \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 4 \\ \hline \end{array}$$

$$= \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} = -\vec{v}_1 + \vec{v}_2$$

$$f = \begin{bmatrix} -1 & -1 \\ & 1 \end{bmatrix}$$

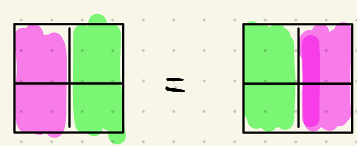
$$f(\vec{v}_2) = -\vec{v}_1 + \vec{v}_2$$

$$f(\vec{v}_1) = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 4 \\ \hline \end{array} = -\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} = -v_1 \qquad \begin{bmatrix} -1 & -1 \\ & 1 \end{bmatrix}$$

## Relations:

$$\begin{array}{|c|c|}\hline a & c \\ \hline b & \\ \hline \end{array} = \begin{array}{|c|c|}\hline c & a \\ \hline b & \\ \hline \end{array} + \begin{array}{|c|c|}\hline a & b \\ \hline c & \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline a & \\ \hline b & \\ \hline \end{array} = -\begin{array}{|c|c|}\hline b & \\ \hline a & \\ \hline \end{array}$$



# Settings in General

Shape of tableaux:

| | | | | |
|---|---|---|---|---|
| | | --- | | |
| | | ... | | |

for $S_d$

Total $d$ boxes

Basis #1: <u>Standard tableaux</u> e.g. $S_8$

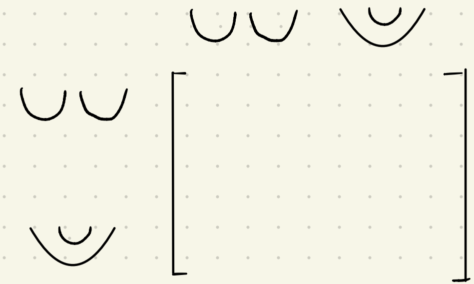
| | | | |
|---|---|---|---|
| 1 | 2 | 4 | 7 |
| 3 | 5 | 6 | 8 |

Basis #2: <u>Noncrossing Webs</u>

e.g.

# Example for $S_4$

$$
\begin{array}{c|c}
 & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \\
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} & \\
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} & \left[\begin{array}{cc} \quad & \quad \\ \quad & \quad \end{array}\right]
\end{array}
$$



$$\begin{bmatrix} -1 & -1 \\ & 1 \end{bmatrix} \neq \begin{bmatrix} -1 & 1 \\ & 1 \end{bmatrix}$$

$$\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \neq \begin{bmatrix} 1 & \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ & 1 \end{bmatrix} \neq \begin{bmatrix} -1 & 1 \\ & 1 \end{bmatrix}$$

# Same Module

$$\mathbb{C}^2 = \text{span}\left\langle \underbrace{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}\ ,\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}}_{\mathcal{B}_1} \right\rangle = \text{span}\langle$$  ,  $\rangle$

$\mathcal{B}_2$

$$\begin{bmatrix} -1 & -1 \\ & 1 \end{bmatrix} \text{ under } \mathcal{B}_1 \rightsquigarrow M \begin{bmatrix} -1 & -1 \\ & 1 \end{bmatrix} M^{-1} \text{ under } \mathcal{B}_2$$

$$M \begin{bmatrix} -1 & -1 \\ & 1 \end{bmatrix} M^{-1} \stackrel{?}{=} \begin{bmatrix} -1 & 1 \\ & 1 \end{bmatrix}$$

$$M \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} M^{-1} \stackrel{?}{=} \begin{bmatrix} 1 & \\ 1 & -1 \end{bmatrix}$$

$$M \begin{bmatrix} -1 & -1 \\ & 1 \end{bmatrix} M^{-1} \stackrel{?}{=} \begin{bmatrix} -1 & 1 \\ & 1 \end{bmatrix}$$

base change

$$M = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$$

# Key Questions

a) Describe M (too hard)

e.g. $m_{ij} = (i-j)^2 - \frac{i}{j} + (-1)^i$ No such formula

b) Describe an algorithm of computing M (Im-Z.)

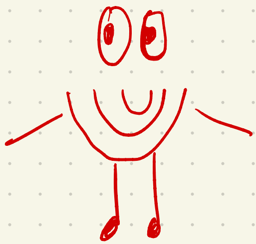
$m_{ij}$ = answer to a COUNTING problem

c) Describe properties of M. (Russell-Tymoczko, Rhoades, Im-Z.)

upper-triangular with nonnegative entries

d) Study $M^{-1}$ (open)

# II. Main Results



# One Example $(d=6)$

## Tableaux Basis

| | | |
|---|---|---|
| 1 | 3 | 5 |
| 2 | 4 | 6 |

| | | |
|---|---|---|
| 1 | 2 | 5 |
| 3 | 4 | 6 |

| | | |
|---|---|---|
| 1 | 3 | 4 |
| 2 | 5 | 6 |

| | | |
|---|---|---|
| 1 | 2 | 4 |
| 3 | 5 | 6 |

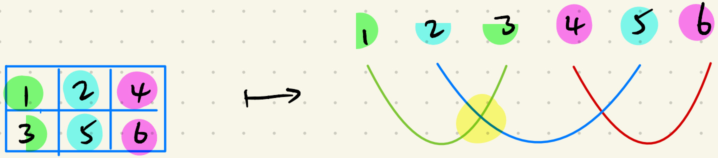
| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |

$\updownarrow$ Base change Matrix $M$

## Web Basis

# Computing One Column

4th Column in M :

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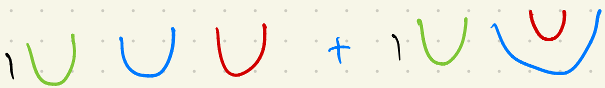
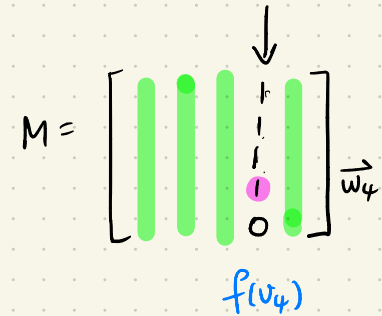
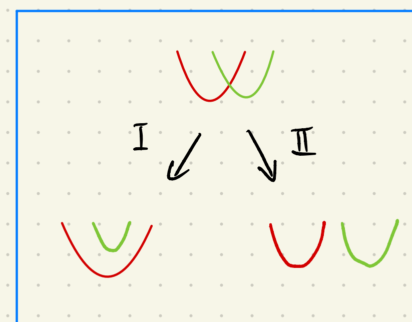
= 



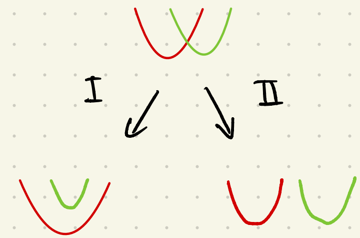

$\vec{w}_4 =$ 



# Computing a Single entry

$$M = \begin{array}{cc} & f(\vec{v}_5) \\ & \downarrow \\ \begin{bmatrix} \quad & & \\ & & \\ & & \bullet \end{bmatrix} & \vec{w}_5 \end{array}$$

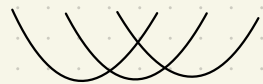
(4,5) - entry ?




$\vec{v}_5 =$

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |

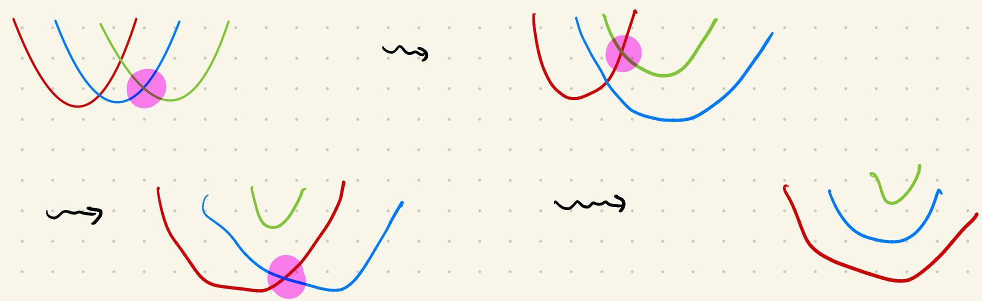
$\longmapsto$



$\overset{?}{\rightsquigarrow}$

$= \vec{w}_5$

Yes !



① = Counting # of ways

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{for } S_6$$

1. Upper-Triangular ( Russell - Tymoczko 2020 )
2. 1's on the diagonal ( Russell - Tymoczko 2020 )
3. Non-negative entries ( Rhoades 2020 )

known

4. Positive entries above diagonal ( Im - Z. 2021 )

# Caveat: Partial Order

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

labeling row: $\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline\end{array}$ $\vec{v}_2$

labeling column: $\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline\end{array}$ $\vec{v}_3$

linear order $2 \leq 3$, but $\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline\end{array} \not\leq \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline\end{array}$ in partial order

(total order)

0 is NOT above diagonal

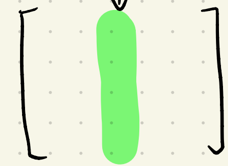
# III. Challenges for You......

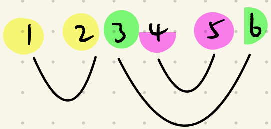
OPEN PROBlEMS

# Inverse Matrix $M^{-1}$

$f^{-1}(\vec{w_3})$

Prop (Im-2.) Algorithm of Computing $M^{-1} =$





$\mapsto$

| | | |
|---|---|---|
| 1 | 3 | 4 |
| 2 | 6 | 5 |

Not Standard !

To resolve .... use Relations:

1.

| | |
|---|---|
| a | c |
| b | |

=

| | |
|---|---|
| c | a |
| b | |

+

| | |
|---|---|
| a | b |
| c | |

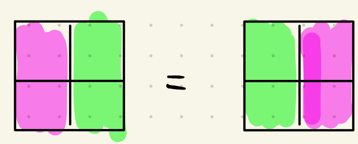
2.

| | |
|---|---|
| a | |
| b | |

= −

| | |
|---|---|
| b | |
| a | |

3.



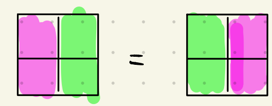
$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 & 5 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 1 & 4 & 3 \\ \hline 2 & 5 & 6 \\ \hline \end{array}$$

$$= \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & 3 & 6 \\ \hline \end{array}$$

$$= \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array} - \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline \end{array}$$

1. $\begin{array}{|c|c|}\hline a & c \\ \hline b & \\ \hline \end{array} = \begin{array}{|c|c|}\hline c & a \\ \hline b & \\ \hline \end{array} + \begin{array}{|c|c|}\hline a & b \\ \hline c & \\ \hline \end{array}$

2. $\begin{array}{|c|c|}\hline a & \\ \hline b & \\ \hline \end{array} = - \begin{array}{|c|c|}\hline b & \\ \hline a & \\ \hline \end{array}$

3. 

Conjecture (IM-2.) The coefficients are $\pm 1$ !

(Challenge !)

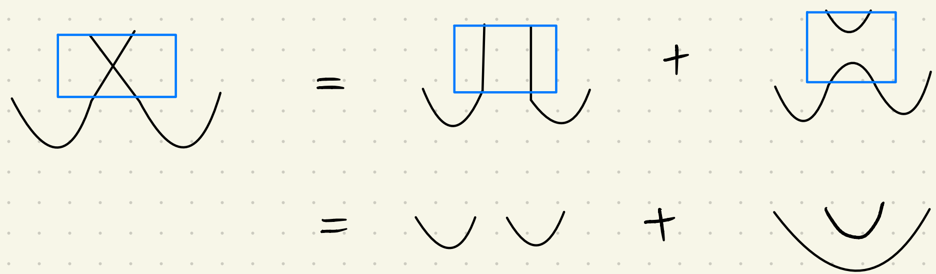
THANK YOU !

# References

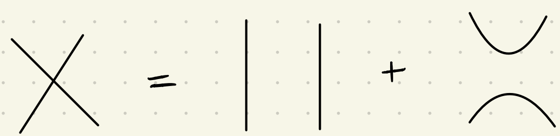

- Heather Russell, Julianna Tymoczko. The transition matrix between the Specht and web bases is unipotent with additional vanishing entries
- Heather Russell, Julianna Tymoczko The transition matrix between the Specht and the $\mathfrak{sl}_3$ web bases is unitriangular w.r.t. shadow containment
- Brendon Rhoades. The web basis expands positively into the polytabloid basis
- Mee Seong Im, Jieru Zhu. Transitioning between the tableaux and web basis for Specht modules.
- Mikhail Khovanov, Greg Kuperberg Web bases for $\mathfrak{sl}(3)$ are not dual canonical

# Adding A Crossing

To two cups

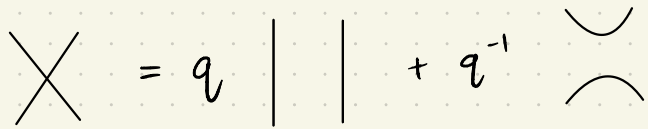


Skein Relation (Jones Polynomials)



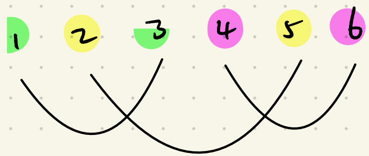
## Quantum Case.

Use




Claim: Entries in M are MONOMIALS !

e.g.




rank ( cup diagram) = # of pairs of nesting arcs

Claim: power of $q$ + 2 · rank = Constant

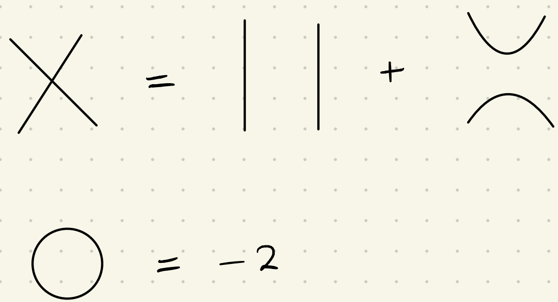
# Computing A Given Row

Prop. ( Im-Z. 2021)

The rows of M can be computed as following

1) associate a web to a tableau

2) expand by :

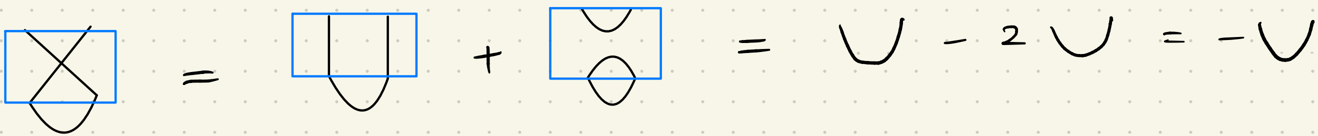


quantum variable
$q \rightsquigarrow 1$

Skein Relation used in Jones polynomial (Fields Medal ! )

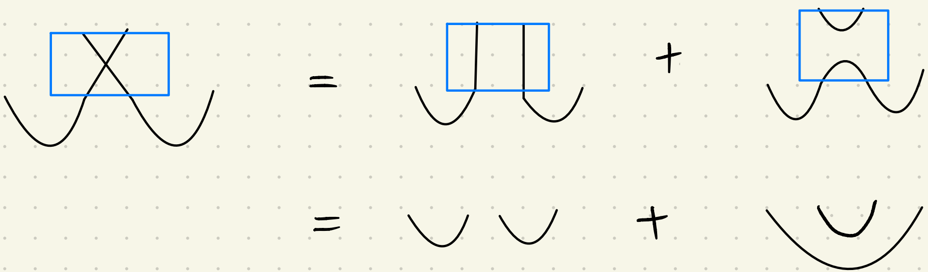
# Adding A Crossing

① To a cup



Each R-1 move produce a negative sign !

② To two cups



# Main Theorems

Theorem ( Im - Z. 2021)

In the upper triangular portion, the relevant count $\geq 1$

i.e a path exists !

Consequence: $M$ is uppertriangular with positive entries

above the diagonal !