Categorifying Type B Schur Algebras

and Beyond... (Cyclotomic)

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Quantum Schur-Weyl Duality

Th (Jimbo)

$$\boxed{U_q(\mathfrak{gl}_n(\mathbb{C})) \twoheadrightarrow \operatorname{End}_{\mathcal{H}_A}(V^{\otimes d}) =: S_n^d \quad \text{Schur algebra}}$$

$q \in \mathbb{C}(q)$ generic, $V = \mathbb{C}(q)^n$

$\mathcal{H}_A$: alg / $\mathbb{C}(q)$ with

generators: $T_1, \ldots\ldots, T_{d-1}$

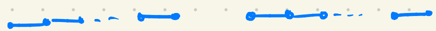
relations $(T_i - 1)(T_i + q^{-1}) = 0$

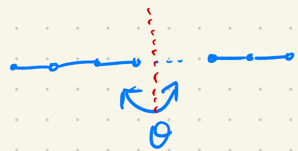
$T_i T_j = T_j T_i \quad |i-j| > 1$

$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad 1 \le i \le d-2$

Type B Schur-Weyl Duality

Type A	$\mathfrak{gl}_n$	Sym. Gp S_d
Type B	$\mathfrak{osp}(m\|2n)$	Brauer Alg. B_d
* Type B	?	$\mathbb{Z}_2 \wr S_d$

①  (w/ J. Kujawa)

②  (w/ Y. Li)

Th (Mazorchuk-Stroppel, Hu-Stoll, Kujawa-Z.) $U(\mathfrak{gl}_{n_1}) \otimes U(\mathfrak{gl}_{n_2}) \longrightarrow \mathrm{End}_{\mathbb{C}B_d}\left(\left(\mathbb{C}^{n_1+n_2}\right)^{\otimes d}\right)$

(Type B Weyl gp)

Classical

Quantum Schur-Weyl Duality (Type B)

Th (Shoji-Sakamoto)

$$\boxed{U_q(\mathfrak{gl}_{n_1}) \otimes U_q(\mathfrak{gl}_{n_2}) \twoheadrightarrow \operatorname{End}_{\mathcal{H}_B}(V^{\otimes d}) =: S^d_{n_1,n_2} \text{ Schur algebra of Type B}}$$

$q \in \mathbb{C}(q, Q_1, Q_2)$ generic, $V = \mathbb{C}(q, Q_1, Q_2)^N$, $N = n_1 + n_2$

alg / $\mathbb{C}(q, Q_1, Q_2)$ with

generators : $T_0, T_1, \dots, T_{d-1}$

relations $(T_i - 1)(T_i + q^{-1}) = 0$

$T_i T_j = T_j T_i \quad |i-j|>1$

$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad 1 \le i \le d-2$

$(T_i - Q_1)(T_i - Q_2) = 0$, $T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0$

Presentation of Schur Algebra

Th (Doty - Giaquito)

S_n^d : is generated by $e_i, f_i, (1 \leq i \leq n-1)$, 1_λ for $\lambda \in \mathbb{Z}^n$,

Subject to relations :

$1_\mu e_i 1_\lambda = 1_\mu 1_{\lambda+\alpha_i} e_i = 0$ unless $\mu = \lambda + \alpha_i$

Lusztig's Quantum Group

$$\left\{ \begin{array}{l} e_i 1_\lambda = 1_{\lambda+\alpha_i} e_i = 1_{\lambda+\alpha_i} e_i 1_\lambda \\ f_i 1_\lambda = 1_{\lambda-\alpha_i} f_i \\ 1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\lambda \end{array} \right.$$

(α_i : root for e_i)

& $1_\lambda = 0$ unless

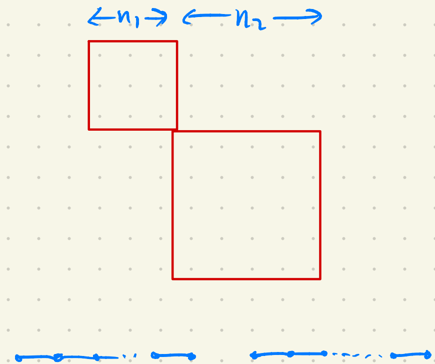
$$\lambda \in \Lambda = \{ \lambda \in \mathbb{Z}^n \mid \lambda_i \geq 0, \ \Sigma \lambda_i = d \}$$

A Presentation for Type B

Th (Kujawa-Z.) $\quad S_{n_1,n_2} \simeq \mathrm{End}_{\mathcal{H}^B_d}\left(\left(\mathbb{C}_q^{n_1+n_2}\right)^{\otimes d}\right) \qquad N = n_1+n_2$

has a presentation with generators $e_i, f_i, \; i \in \{1,\dots,N\} \setminus \{n_1\}$,

$1_\lambda, \; \lambda \in \mathbb{Z}^N$



Subject to relations

$$e_i 1_\lambda = 1_{\lambda+\alpha_i} e_i$$

Same as before

$$f_i 1_\lambda = 1_{\lambda-\alpha_i} f_i$$

$$1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\lambda$$

$1_\lambda = 0$ unless

$$\lambda \in \Lambda = \{\lambda \in \mathbb{Z}^N \mid \lambda_i \geq 0, \; \Sigma \lambda_i = d\}$$

A Categorification in Type A

KLR 2-category $\mathcal{U}$: graded $\mathbb{Z}[q,q^{-1}]$ - linear,

objects $\lambda \in \mathbb{Z}^n$

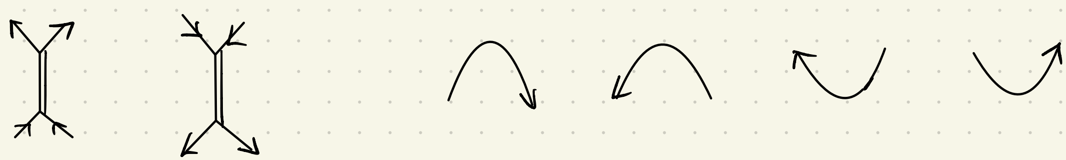
1-mor: 1_λ = identity in $\mathrm{Hom}(\lambda,\lambda)$

$1_{\alpha+\lambda} e_\alpha 1_\lambda \in \mathrm{Hom}(\lambda, \alpha+\lambda)$

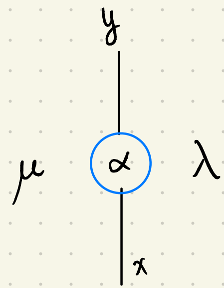
$\mathrm{Hom}(\lambda,\mu)$ = $\mathbb{Z}$-span of words in the form of

$1_\mu e_{\pm i_1} \cdots e_{\pm i_s} 1_\lambda$ $\quad (e_{-i} = f_i)$

2-mor: Vertical & horizontal stacking of



2-Categories in Drawing



λ, μ: objects

x, y: 1-morphisms

α: 2-morphism

$\mathrm{Kar}(\dot{\mathcal{U}})$: idempotent completion of $\dot{\mathcal{U}}$

$$K_0(\mathcal{C}) = \mathbb{Z}[q,q^{-1}]\langle \text{isoclasses of } 1\text{-mor} \rangle / \sim .$$

$\sim$: $0 \to A \to B \to C \to 0$ exact $\Rightarrow$ $[B] = [A] + [C]$

$q \curvearrowright K_0(\mathcal{C})$ grade shift

Th (Khovanov-Lauda, Rouquier)

$$K_0(\mathrm{Kar}(\dot{\mathcal{U}})) \cong U_q(\mathfrak{sl}_n)$$

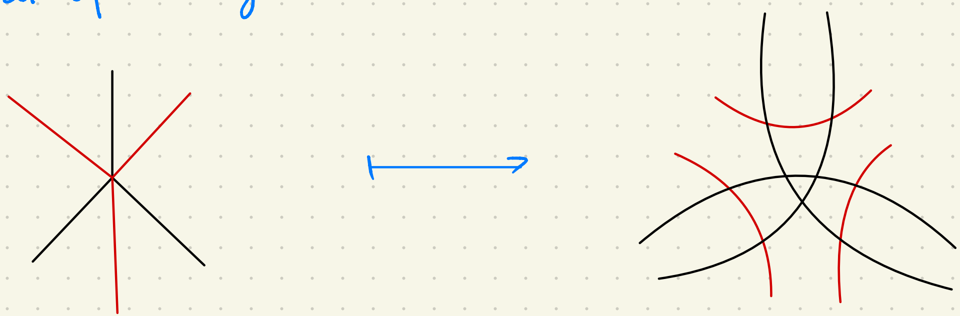
"Schur Category" . $S \simeq \dot{\mathcal{U}}/\sim$ $\qquad \boxed{\lambda} = 0 \quad \lambda \notin \Lambda$

Th (Mackaay – Stosic – Vaz)

$$K_0(Kar(S)) \simeq S_n^d$$

$$\mathcal{H}_A \hookrightarrow S_n^d$$

Cat of Soergel bimods a.k.a Hecke 2-cat.



Type B

Work-in-Progress (Kujawa - Z.)

$\Gamma =$ 

$\Lambda = \{ \lambda \in \mathbb{Z}_{\geq 0}^{n_1+n_2} \mid \Sigma \lambda_i = d \}$

$S(\Gamma)$ categorifies Type B Schur alg.

① $S^d_{n_1,n_2} \simeq \mathrm{End}_{\mathcal{H}^B_d} \left((\mathbb{C}_q^{n_1+n_2})^{\otimes d} \right)$ first kind

action

② (Ariki) $\mathrm{End}_{\mathcal{H}} \left(\bigoplus_\mu x_\mu \mathcal{H} \right)$ second kind

③ (Lai-Nakano-Xiang) $\mathrm{End}_{\mathcal{H}^B_d} \left((\mathbb{C}_q^{n_1+n_2})^{\otimes d} \right)$

different action

$\simeq$ ① under Separation Condition

Alternative Type B

Th (Bao-Kujawa-Li-Wang)

Quantum

$$\mathcal{U}^\imath \twoheadrightarrow \mathrm{End}_{\mathcal{H}_d^B}(V^{\otimes d})$$

or

$$\mathcal{U}^\jmath \twoheadrightarrow \mathrm{End}_{\mathcal{H}_d^B}(V^{\otimes d})$$

$\boxed{\begin{array}{l} \mathcal{U}^\imath \subseteq U_q(\mathfrak{gl}_{2n}) \\ V = \mathbb{Q}(\sqrt{q})^{\otimes 2n} \end{array}}$

$\boxed{\begin{array}{l} \mathcal{U}^\jmath \subseteq U_q(\mathfrak{gl}_{2n+1}) \\ V = \mathbb{Q}(\sqrt{q})^{\otimes 2n+1} \end{array}}$

D- step flags $(D = 2d)$

$$\mathcal{F} = \{ \{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_D \subseteq \mathbb{F}_q^N \}$$

isotropic : $V_i^\perp = V_{N-i}$ w.r.t sym. bil. form

$$\mathcal{U}^\imath \simeq \text{orbit functions on } \mathcal{F} \times \mathcal{F} / O_N$$

θ graph automorphism.

$$\theta: \quad U(\mathfrak{gl}_N) \to U(\mathfrak{gl}_N) \quad \text{involution}$$

$$\mathcal{U}(\mathfrak{gl}_N)^{\theta} \hookrightarrow \mathcal{U}(\mathfrak{gl}_N),$$

Prop. (Li-Z.) $\mathcal{U}(\mathfrak{gl}_N)^{\theta} \twoheadrightarrow \mathrm{End}_{\mathcal{B}_d}((\mathbb{C}_q^{2n})^{\otimes d})$

(Lezter - Kolb) presentation for $\mathcal{U}_q(\mathfrak{gl}_N)^{\theta}$

Isomorphisms

Th (Li-Z.) $\exists$ base change $V^{\otimes d} \to V^{\otimes d}$ (2 : invertible)

$\rightsquigarrow$ " separation condition"

$(V = \mathbb{C}^{2n})$ $v_i \mapsto v_i + v_{\bar{i}}$

$v_{\bar{i}} \mapsto v_i - v_{\bar{i}}$

$(V = \mathbb{C}^{2n+1})$ & $v_0 \mapsto v_0$

$$\begin{tikzcd}
\mathcal{U}(\mathfrak{gl}_n \oplus \mathfrak{gl}_n) \arrow[r, two heads] \arrow[d, "\simeq"'] & \mathrm{End}_{B_d}(V^{\otimes d}) \arrow[d, "\simeq"] \\
\mathcal{U}(\mathfrak{gl}_{2n})^{\theta} \arrow[r, two heads] & \mathrm{End}_{B_d}(V^{\otimes d})
\end{tikzcd}$$

$B_d \nearrow$ action

$B_d \nearrow$ different action

Quantum case : Missing $U_q(\mathfrak{gl}_n \oplus \mathfrak{gl}_n) \not\simeq \mathcal{U}^{\imath}$ $\leftarrow$ not naturally a Hopf alg.

Alternative Presentation

Th (Li-Z.)

$$S^{d}_{n_1,n_2} \simeq U(\mathfrak{gl}_N)^{\theta} / \sim \qquad |n_1 - n_2| \leq 1$$

$\sim$ further relations

(Bao-Kujawa-Li-Wang)

Conjectured full list of relations in quantum setting

E.g.

$n=2$

$d=3$

$$([e,f]+d_1-0)([e,f]+d_1-2)\ \cdots\ ([e,f]+d_1-6)=0$$

missing: Geometric proof in quantum setting