THE SL_2 TILTING CATEGORY

in the mixed case

HIM
Junior
Trimester
Program Fall 2020

Louise Sutton
(OIST)

Daniel Tubbenhauer
(Sydney)

Paul Wedrich
(Hamburg)

Jieru Zhu * (OIST)

Quasihereditary Algebra

Highest weight Category $\mathcal{C}$ (Cline - Parshall - Scott)

- Λ : poset $\forall$ interval : finite
- $\lambda \in \Lambda$ $L(\lambda)$: simple. $P(\lambda)$ Projectives

  $\Delta(\lambda)$ standard object $\nabla(\lambda)$ costandard object
- $\mathrm{End}(L(\lambda)) = \Bbbk$ ← Schur's Lemma
- $L(\lambda) \to \nabla(\lambda)$ inj hull in $\mathcal{C}_{\leq\lambda}$, $\ker \subseteq \mathcal{C}_{<\lambda}$

  $\Delta(\lambda) \twoheadrightarrow L(\lambda)$ proj cover in $\mathcal{C}_{\leq\lambda}$,
- $\mathrm{Ext}^2(\Delta(\lambda), \nabla(\mu)) = 0$ ←

Tilting category

T : tilting $\iff$ T has a

$W_{i+1}/W_i \simeq \Delta(\lambda)$

filtraction $0 \subseteq W_1 \subseteq W_2 \subseteq \dots \subseteq W_n = T$

standard flag : subquotients are Δ's

AND costandard flag : subquotients are ∇'s

Properties

$\forall \lambda$ $\exists$ unique tilting $\underline{T(\lambda)}$, $L(\lambda)$, $P(\lambda)$, $\underline{\Delta(\lambda)}$, $\nabla(\lambda)$

The Tilting Category : ←

"Tilt" full sub cat of $\mathcal{C}$ gen. $(\oplus)$ by $T(\lambda)$

Sl_2-mod over $\mathbb{C}$ ← alg closed, char 0

$V = \mathbb{C}^2$ natural rep. $\quad V = L(\varepsilon_1)$

$T^\bullet(V) = \bigoplus_{i=0}^{\infty} V^{\otimes i}$

$S^\bullet(V) := T^\bullet(V) / \langle v_1 \otimes v_2 - v_2 \otimes v_1 \rangle$

Weyl module

$\Delta(n) = \mathrm{Sym}^n(V)$: n-th graded piece $S^\bullet(V)$

Also: $\quad T(n) = \Delta(n)$ $\quad$ highest weight $n\varepsilon_1$ $\quad \Lambda = \mathbb{Z}_{\geq 0}$

$\Lambda = \mathbb{Z}_{\geq 0}$

The Tilting Category is **Semisimple**

$\underline{\Delta(m) \otimes \Delta(n)} \simeq \underline{\bigoplus \Delta\text{'s}}$

$V = S^1(V)$

$\Delta(m) \otimes \Delta(1) = \Delta(m+1) \oplus \Delta(m-1)$

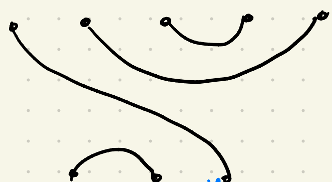
$V^{\otimes n} = \Delta(1)^{\otimes n} \supseteq \Delta(n)$

$T(m) = \Delta(m)$ summand of $V^{\otimes m}$

The Temperley-Lieb category

- **object**: $n \in \mathbb{Z}_{\geq 0}$
- **morphisms**: noncrossing matchings

e.g. $\quad \mathrm{Hom}(3,5)$



"standard basis"

the full on subcategory generated by V

$$\text{TL Cat} \xrightarrow[\text{monoidal}]{\mathcal{F}} Sl_2\text{-Mod}$$

any morphism $\mathrm{Hom}(V^{\otimes n_1}, V^{\otimes n_2})$ can be represented diagrantically

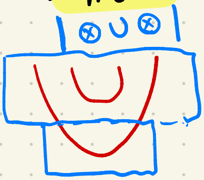
vertical stacking $\mapsto$ composition of morphism

$\boxed{g}$ over $\boxed{f}$ $\mapsto g \circ f$

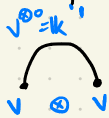
horizontal stacking

$\boxed{g}\ \boxed{f} \mapsto g \otimes f$

- **obj** $n \longmapsto V^{\otimes n}$
- **mor**



$|\ \longmapsto\ id : V \to V$ sl_2



$\longmapsto$ ev : $V \otimes V \simeq V^* \otimes V \to \Bbbk$

$f \otimes v \mapsto f(v)$

$\longmapsto$ coev : $\Bbbk \to \underbrace{V \otimes V^*} \simeq V \otimes V$

$1 \mapsto v_1 \otimes v_1^* + v_2 \otimes v_2^*$

$= 2 = \dim V$

$q + q^{-1} = \operatorname{qdim} V_q$

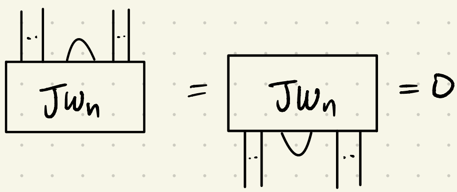
Jones-Wenzl Idempotents

$$JW_n : \underset{V^{\otimes n}}{\Delta(1)^{\otimes n}} \xrightarrow{proj}\!\!\!\!\rightarrow \Delta(n) \hookrightarrow \overset{V^{\otimes n}}{\Delta(1)^{\otimes n}} \in TL(n,n)$$

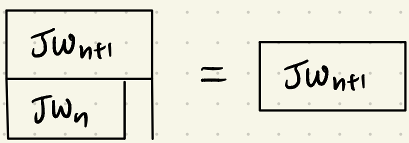
$V^{\otimes n} = \Delta(1)^{\otimes n} \supseteq \Delta(n) = T(n)$

$\{JW_n\}_{n=1}^{\infty}$ **uniquely determined** by

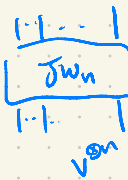
- idempotent $Hom(\Delta(n), V^{\otimes(n-2)}) = 0$
- "uncappable"



- absorbtion



$JW_n \in TL(n,n)$



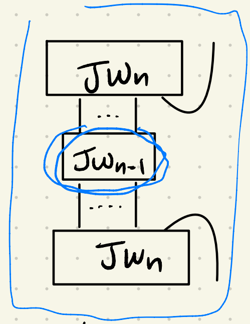
$$V^{\otimes n+1} \longrightarrow \Delta(n) \otimes V \xrightarrow[\text{summand}]{} \!\!\!\!\rightarrow \Delta(n+1)$$

Recursive Formula

over $\mathbb{C}$

$\Delta(n) \otimes \Delta(1) \simeq \Delta(n+1) \oplus \Delta(n-1)$

$$\rightarrow \quad \boxed{JW_n}\,\Big| \;=\; \boxed{JW_{n+1}} \;-\; \frac{n}{n+1}$$



$$\Delta(n) \otimes \Delta(1) \simeq \Delta(n+1) \oplus \Delta(n-1)$$

Pieri rule

In Positive Characteristic

char $\Bbbk = p$ $\qquad SL_2(\Bbbk)$ - mod

$T(n)$: no longer standard.

$$\text{supp}(n) = \{ m \mid \Delta(m) : \text{subquotient of } T(n) \}$$

e.g. $\quad p = 5$

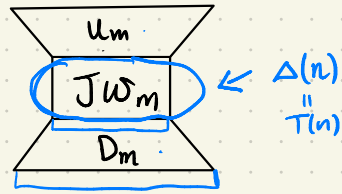
$$n+1 = |2\,1\,4\,0\,3|_5 \qquad m+1 = |2\ {-1}\ 4\ 0\ {-3}|_5$$

$m \in \text{supp}(n)$ $\qquad$ "all digit flips"

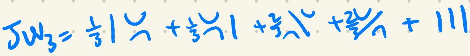
p- Jones Wenzl projectors

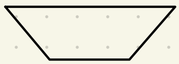
(Burrull - Libedinsky - Sentinelli)

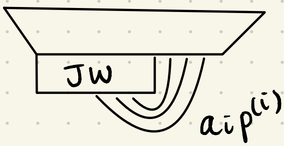
$$JW_n^p := \underset{\text{in char } p}{JW_n} + \sum_{\substack{m \in supp(n) \\ m \neq n}} \underbrace{c_m}$$

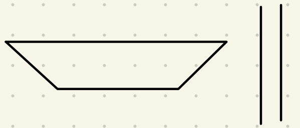


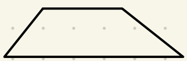
$n=3$

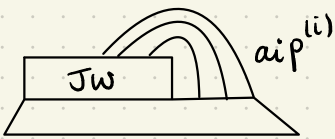


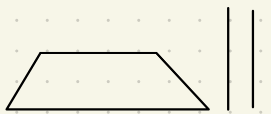
$U_m :=$  built from





$D_m =$  built from





Theorem (Burrull - Libedinsky - Sentinelli)

$\boxed{JW_n^p}$

- well-defined over $\mathbb{F}_p$
- idempotent
- satisfy the absorption properties of JW's 
- decomposition into pairwise orthogonal idempotents
- categorifies the KL-basis in char p

$$\text{two colored TL} \xrightarrow{\mathcal{F}} \text{Soergel bimodules}$$

$$\boxed{JW_n^p} \longmapsto \text{indecomposable obj}$$

(Tubbenhauer - Wedrich)

Ringel Duality

$$V^{\otimes n}$$

$$\mathrm{Im}\left(\boxed{JW_n^p} \right) \cong T(n)$$

$$\underbrace{\qquad}_{n} \qquad V^{\otimes n}$$

$$V^{\otimes n} \xrightarrow{\text{proj}} T(n) \xhookrightarrow{\text{inclusion}} V^{\otimes n}$$

$$T(n) \subseteq_{\text{summand}} V^{\otimes n}$$

Quiver for sl_2 - tilting category (char $\Bbbk > 0$)

Diagrammatically

Tilt

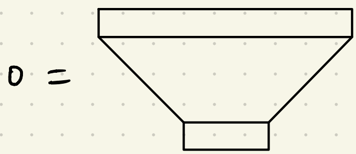
- obj: $n \in \mathbb{Z}_{\geq 0}$
- mor: pictures which factor through

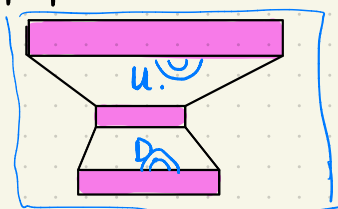
$\mathrm{Hom}(T(n), T(m))$

$\boxed{JW_n^P} \leftrightarrow T(n)$

$n \in \mathbb{Z}_{\geq 0}$

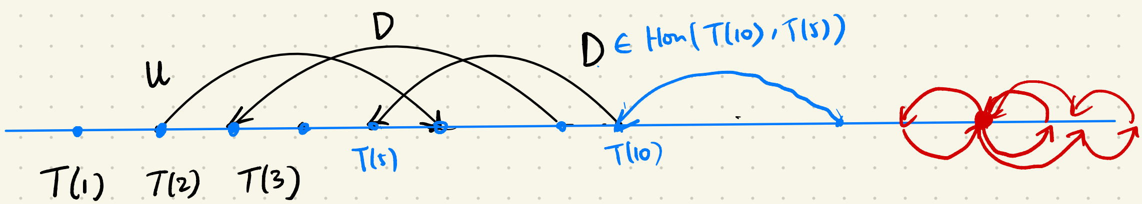
"normal" JW's "purple" JW's





"cappable at certain positions"

(Tubbenhauer - Wedrich)



- Full list of relations among morphisms in Tilt

e.g. $DD = 0$ $UU = 0$

$DU = \underline{fUD + gUUDD}$ Complete!

- generators: D, U arrows
- relations

DUDUUD

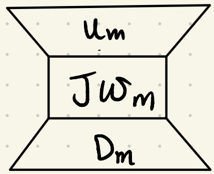
(Sutton - Tubbenhauer - Wedrich - Z.)

char $\Bbbk = p$, $\quad q^{2\ell} = 1$ $\quad \ell$ smallest, $q \in \Bbbk$

The (p, ℓ)-adic exp. $\quad |a_k, \dots, a_0|_{p,\ell} = \sum_{i=0}^{k} a_i p^{(i)}$

$$p^{(0)} = 1, \qquad p^{(i)} = p^{i-1}\ell \qquad i \geq 1$$

$$\boxed{JW_n^{p,\ell}} := \boxed{JW_n} + \sum_{\substack{m \in \mathrm{supp}(n) \\ m \neq n}} [c_m]_q$$



$\mathrm{supp}(n) = \{ m \mid \Delta(m) :$ subquotients of $T(n) \}$

$= \{ m \mid m+1$ "digit flip" of $n+1$ in (p, ℓ) - adic expansion $\}$

Theorem (Sutton - Tubbenhauer - Wedrich - Z.)

$\boxed{JW_n^{p,\ell}}$ well-defined over $\Bbbk$, absorption property

idempotent, orthogonal decomp, $V^{\otimes n} \longrightarrow T(n) \hookrightarrow V^{\otimes n}$

& similar relations as in Tubbenhauer - Wedrich

13:40 Wed 18. Aug

arxiv.org

91 %

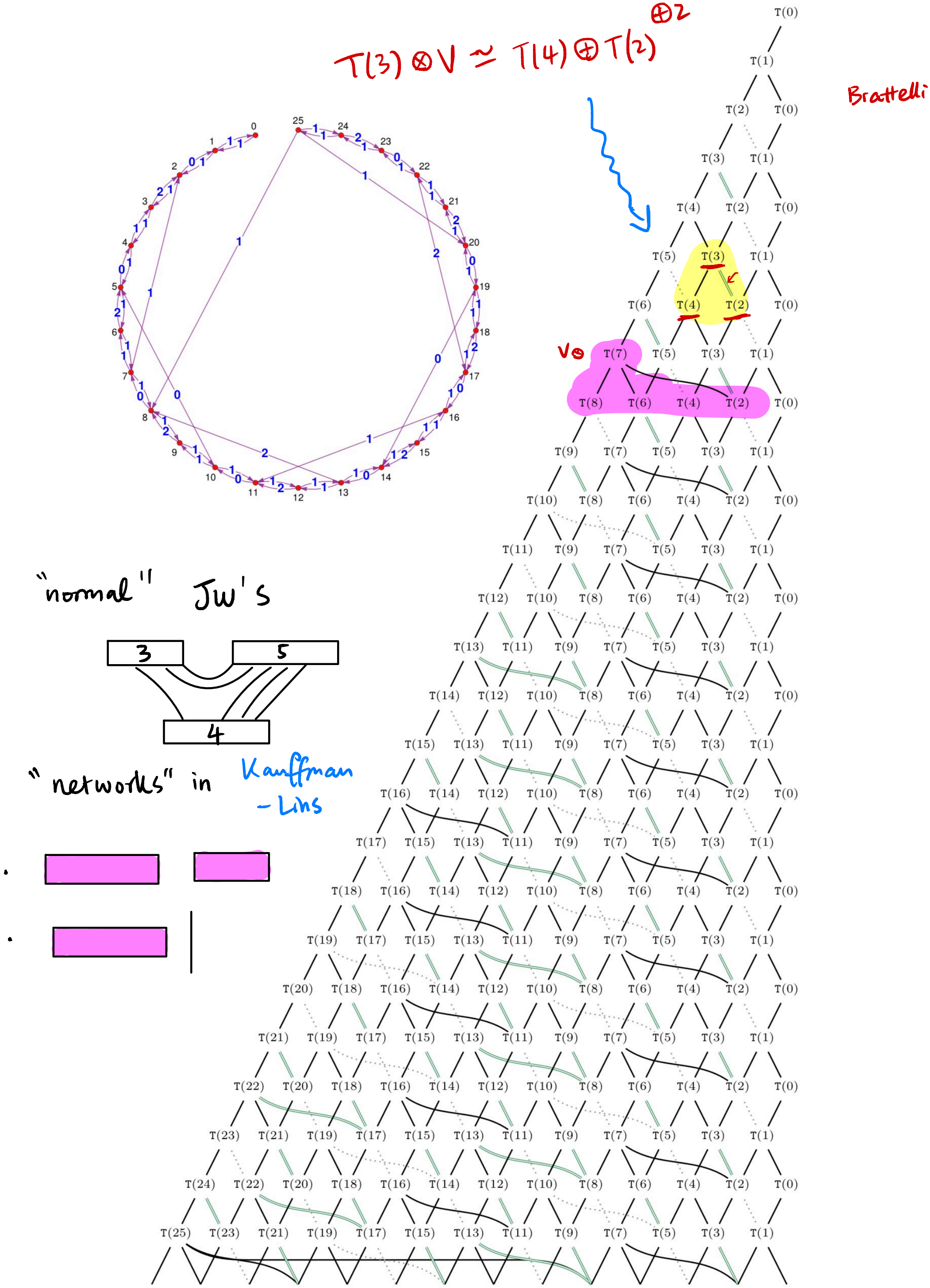
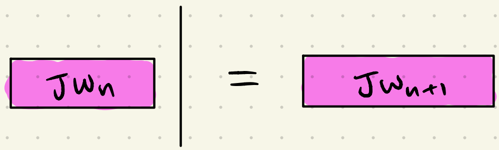


FIGURE 3. *Top left picture:* The full subgraph of the fusion graph for T(1) in

Theorem (Sutton-Tubbenhauer-Wedrich-Z.)

$$n+1 = |\ast\ast\cdots\ast\, \underline{a_2\ a_1\ a_0}|_{p,\ell}$$

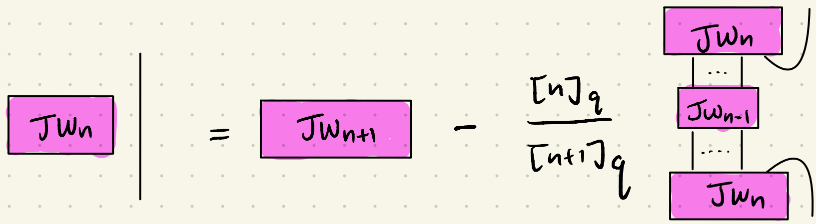
① $T(n)\otimes V \simeq T(n+1)$ if $a_0 = 0$



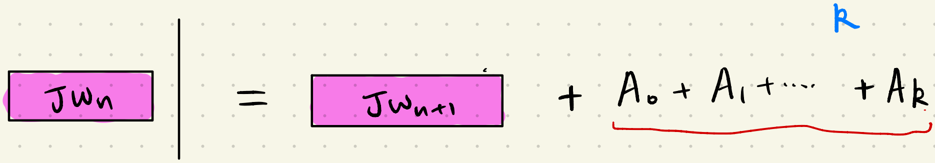
② otherwise

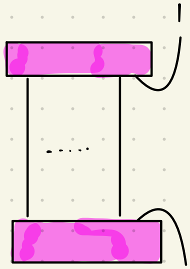
$T(n)\otimes V \simeq T(n+1) \oplus T(n-1)$

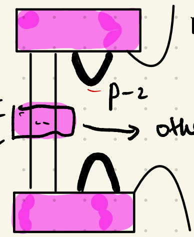
"normal recursion in char 0"



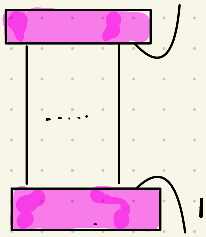
③ if $a_0 = \ell - 1$, $n+1 = |\ast\ast\cdots\ast\, \underbrace{p\text{-}1\ \ p\text{-}1\ \cdots\ p\text{-}1\ \ \ell\text{-}1}_{k}|_{\ell,p}$

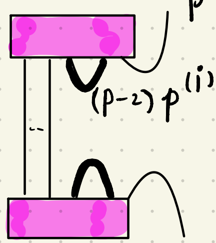


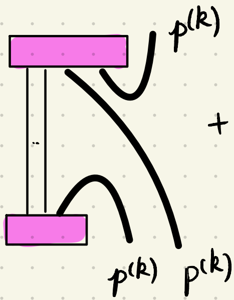
$A_0 = \frac{1}{g}$ 

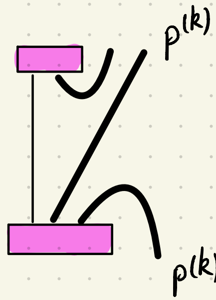
$- \frac{f}{g}$ 

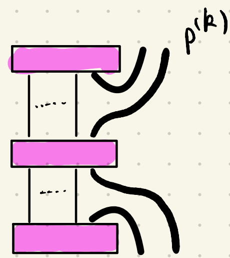
$\longrightarrow$ other T(n) in the decomposition of $T(n) \otimes V$

$A_i = \frac{1}{g}$ 

$- \frac{f}{g}$ 

$A_k =$ 

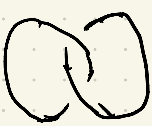
$+$ 

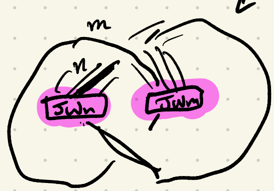
$+ \ 2$ 

(sometimes)

AND A_i : pairwise orthogonal idempotents

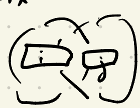
$$T(n) \otimes V \twoheadrightarrow T(m) \longrightarrow T(n) \otimes V$$





Kauffman-Lins

S matrix

$[S_{ij}] =$ 

$T(s) \subset T(n) \otimes T(m)$