THE SL_2 TILTING CATEGORY

in the mixed case

HIM
Junior
Trimester
Program Fall 2020

2021 SIAM

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Highest weight Category $\mathcal{C}$ (Cline - Parshall - Scott)

- Λ : poset $\forall$ interval : finite
- $\lambda \in \Lambda$ $L(\lambda)$: simple.

  $\Delta(\lambda)$ standard object $\nabla(\lambda)$ costandard object
- $\mathrm{End}(L(\lambda)) = \Bbbk$
- $L(\lambda) \to \nabla(\lambda)$ inj hull in $\mathcal{C}_{\le\lambda}$, ker $\subseteq \mathcal{C}_{<\lambda}$

  $\Delta(\lambda) \to L(\lambda)$ proj cover in $\mathcal{C}_{\le\lambda}$,
- $\mathrm{Ext}^2(\Delta(\lambda), \nabla(\mu)) = 0$

Tilting category

T : tilting $\Longleftrightarrow$ T has a

standard flag : subquotients are Δ's

AND costandard flag : subquotients are ∇'s

Properties

$\forall \lambda$ $\exists$ unique tilting $T(\lambda)$

The Tilting Category :

"Tilt" full sub cat of $\mathcal{C}$ gen. $(\oplus)$ by $T(\lambda)$

Sl_2-mod over $\mathbb{C}$

$V = \mathbb{C}^2$ natural rep. $\qquad V = L(\varepsilon_1)$

Weyl module

$\Delta(n) = \mathrm{Sym}^n(V) := V^{\otimes n} / \langle v_1 \otimes v_2 - v_2 \otimes v_1 \rangle$, $\quad V = \Delta(1)$

Also: $T(n) = \Delta(n)$ highest weight $n\varepsilon_1$ $\qquad \Lambda = \mathbb{Z}_{\geq 0}$

The Tilting Category is **semisimple**

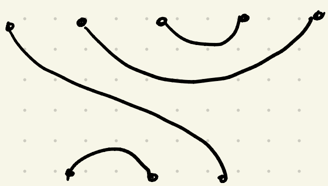
$$\Delta(m) \otimes \Delta(n) \simeq \bigoplus \Delta\text{'s}$$

$\Delta(m)$ summand of $V^{\otimes m}$

The Temperley-Lieb category

- object: $n \in \mathbb{Z}_{\geq 0}$
- morphisms: noncrossing matchings

e.g. $\mathrm{Hom}(3,5)$



"standard basis"

$$\text{TL Cat} \xrightarrow{\mathcal{F}} Sl_2\text{-Mod}$$

- **obj** $\quad n \longmapsto V^{\otimes n}$
- **mor** $\quad | \longmapsto \mathrm{id} : V \to V$

 $\longmapsto$ **ev** : $V \otimes V \simeq V^* \otimes V \to \Bbbk$

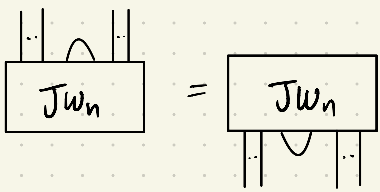
 $\longmapsto$ **coev** : $\Bbbk \to V \otimes V^* \simeq V \otimes V$

Jones-Wenzl Idempotents

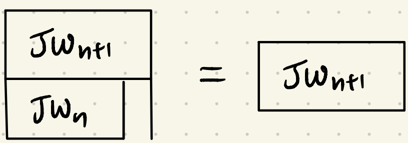
$JW_n : \quad \Delta(1)^{\otimes n} \twoheadrightarrow \Delta(n) \hookrightarrow \Delta(1)^{\otimes n} \in TL(n,n)$

$\{JW_n\}_{n=1}^{\infty}$ **uniquely determined** by

- idempotent $\qquad \mathrm{Hom}(\Delta(n), V^{\otimes(n-2)}) = 0$
- "uncappable"

 $= 0$

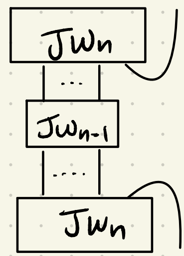
- absorbtion



$$V^{\otimes n+1} \longrightarrow \Delta(n) \otimes V \underset{\text{summand}}{\twoheadrightarrow} \Delta(n+1)$$

Recursive Formula

over $\mathbb{C}$

$$\boxed{JW_n}\;\Big| \;=\; \boxed{JW_{n+1}} \;-\; \frac{n}{n+1}$$ 

$$\Delta(n) \otimes \Delta(1) \simeq \Delta(n+1) \quad \oplus \quad \Delta(n-1)$$

Pieri rule

In Positive Characteristic

$\operatorname{char} \Bbbk = p$ $\qquad SL_2(\Bbbk)$-mod

$T(n)$: no longer standard.

$$\text{supp}(n) = \{ m \mid \Delta(m) : \text{subquotient of } T(n) \}$$

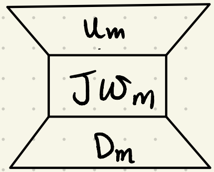
e.g. $\quad p = 5$

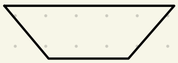
$$n+1 = |2\,1\,4\,0\,3|_5 \qquad m+1 = |2\;{-1}\;4\;0\;{-3}|_5$$

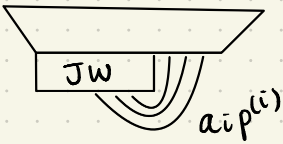
$m \in \text{supp}(n)$ $\qquad$ "all digit flips"

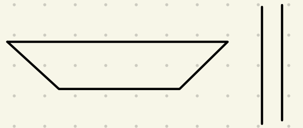
p-Jones Wenzl projectors

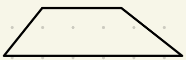
(Burrull-Libedinsky-Sentinelli)

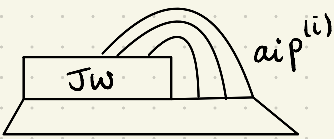
$$\boxed{JW_n^p} := \boxed{JW_n} + \sum_{\substack{m \in supp(n) \\ m \neq n}} c_m$$ 

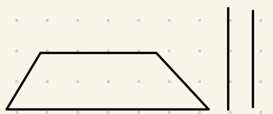
$U_m :=$  built from





$D_m =$  built from





Theorem (Burrull - Libedinsky - Sentinelli)

$\boxed{JW_n^p}$

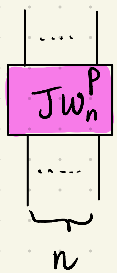
- well-defined over $\mathbb{F}_p$
- idempotent
- Satisfy the absorption properties of JW's
- decomposition into pairwise orthogonal idempotents
- Categorifies the KL-basis in char p

$$\text{two colored TL} \xrightarrow{\mathcal{F}} \text{Soergel bimodules}$$

$$\boxed{JW_n^p} \longmapsto \text{indecomposable obj}$$

(Tubbenhauer - Wedrich)

Ringel Duality



$$V^{\otimes n} \longrightarrow T(n) \longleftrightarrow V^{\otimes n}$$

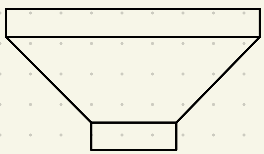
Quiver for sl_2 - tilting category (char $\Bbbk > 0$)

Diagrammatically

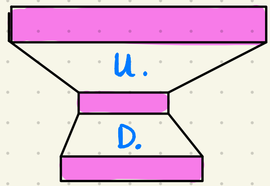
Tilt

- obj: $n \in \mathbb{Z}_{\geq 0}$
- mor: pictures which factor through $\boxed{JW_n^P}$ $n \in \mathbb{Z}_{\geq 0}$

"normal" JW's

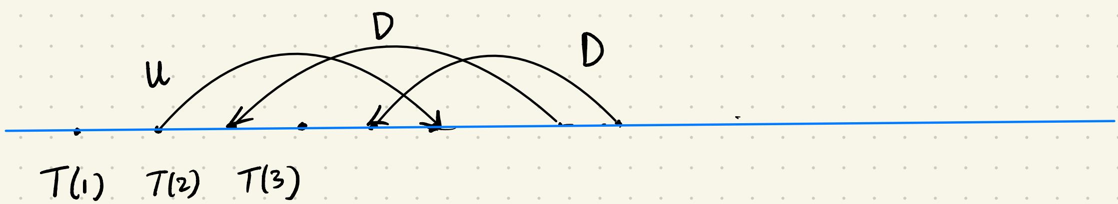
$0 =$ 

"purple" JW's



"cappable at certain positions"

(Tubbenhauer - Wedrich)



- Full list of relations among morphisms in Tilt

e.g. $DD = 0 \qquad UU = 0$

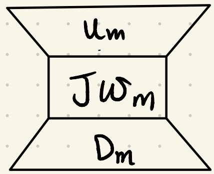
$$DU = fUD + gUUDD$$

(Sutton - Tubbenhauer - Wedrich - Z.)

char $\Bbbk = p$, $\quad q^{2\ell} = 1$ $\quad \ell$ smallest, $q \in \Bbbk$

The (p,ℓ)-adic exp. $\quad [a_k, \dots, a_0]_{p,\ell} = \sum_{i=0}^{k} a_i p^{(i)}$

$$p^{(0)} = 1, \qquad p^{(i)} = p^{i-1}\ell \qquad i \geq 1$$

$$\boxed{JW_n^{p,\ell}} := \boxed{JW_n} + \sum_{\substack{m \in \mathrm{supp}(n) \\ m \neq n}} [C_m]_q$$ 

$\mathrm{supp}(n) = \{ m \mid \Delta(m) :$ subquotients of $T(n) \}$

$= \{ m \mid m+1$ "digit flip" of $n+1$ in (p,ℓ)-adic expansion $\}$

Theorem (Sutton - Tubbenhauer - Wedrich - Z.)

$\boxed{JW_n^{p,\ell}}$ well-defined over $\Bbbk$, absorption property

idempotent, orthogonal decomp, $V^{\otimes n} \longrightarrow T(n) \hookrightarrow V^{\otimes n}$

& similar relations as in Tubbenhaur - Wedrich

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$T(3) \otimes V \simeq T(4) \oplus T(2)$ $\oplus 2$

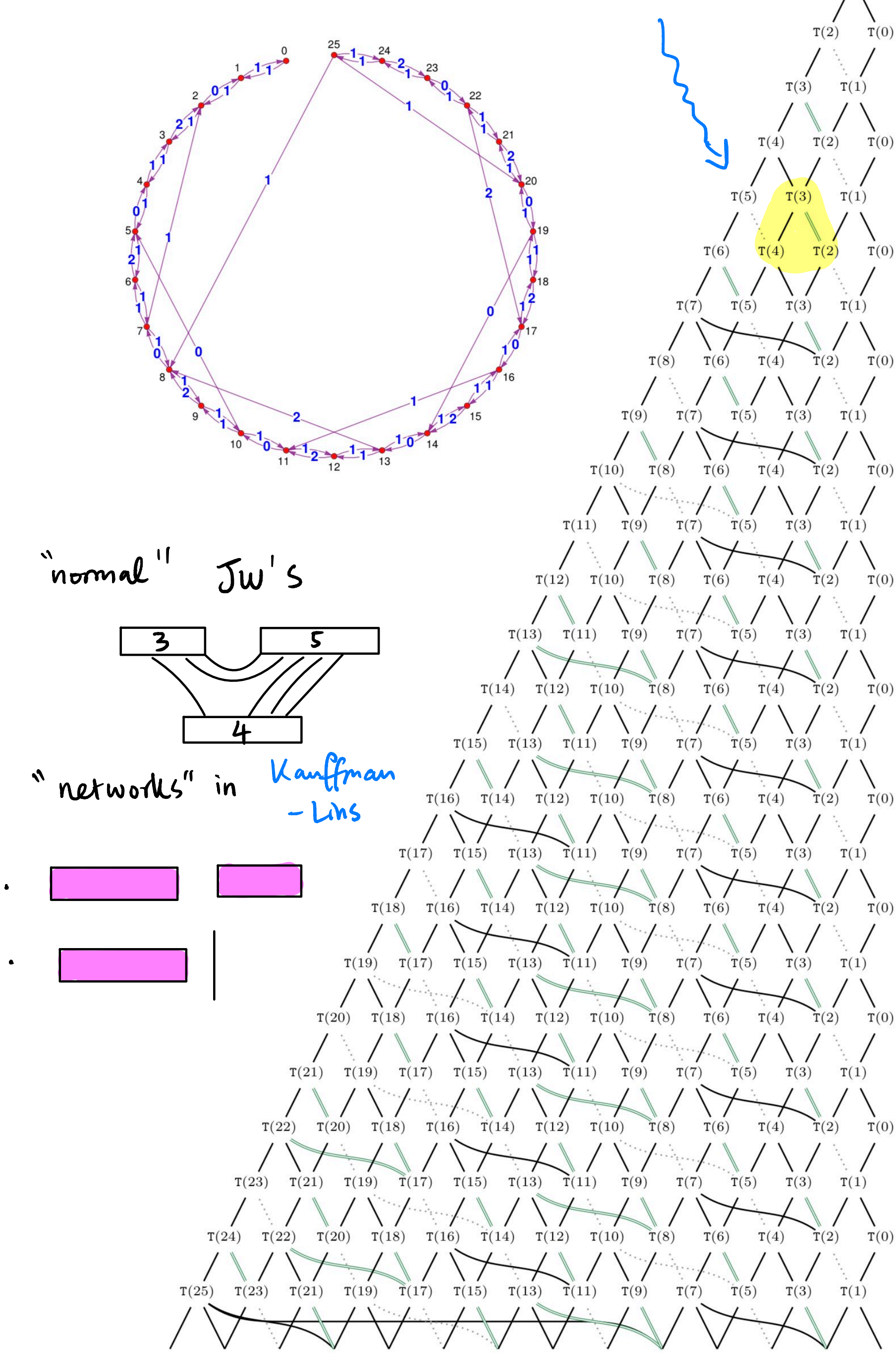
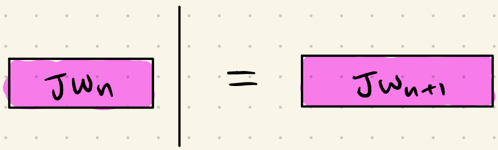


FIGURE 3. *Top left picture:* The full subgraph of the fusion graph for T(1) in

Theorem (Sutton–Tubbenhauer–Wedrich–Z.)

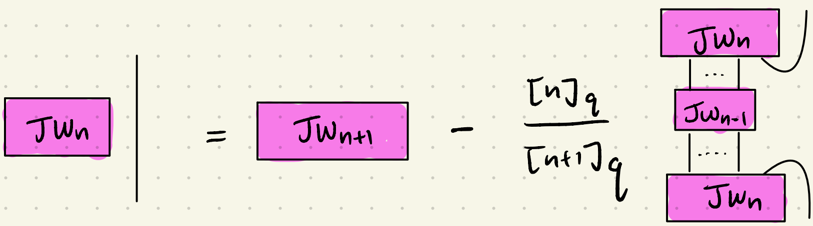
$$n+1 = |\ast\ast\cdots\ast\, a_2\, a_1\, a_0|_{p,\ell}$$

① $T(n)\otimes V \simeq T(n+1)$ if $a_0 = 0$

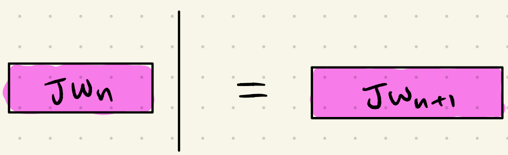


② otherwise

$$T(n)\otimes V \simeq T(n+1)\oplus T(n-1)$$

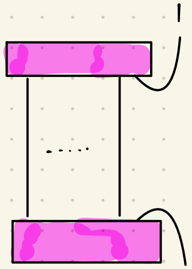


③ if $a_0 = \ell-1$, $n+1 = |\ast\ast\cdots\ast\, \underbrace{p\text{-}1\ \ p\text{-}1\ \cdots\ p\text{-}1\ \ \ell\text{-}1}_{k}|_{\ell,p}$

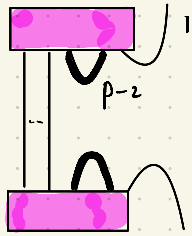


$$+\ A_0 + A_1 + \cdots + A_k$$

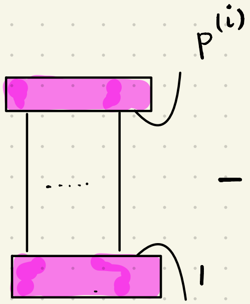
$A_0 = \frac{1}{g}$



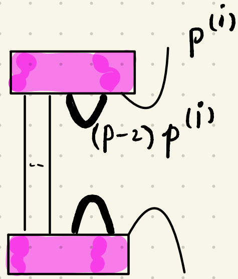
$- \frac{f}{g}$



$A_i = \frac{1}{g}$

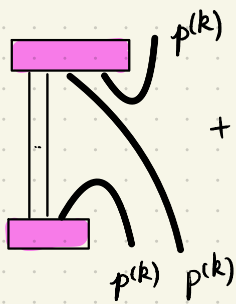


$- \frac{f}{g}$

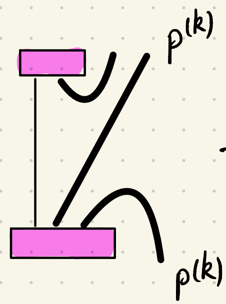


$A_k =$

(sometimes)



$+$



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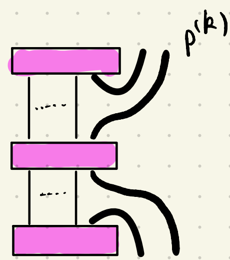

AND A_i : pairwise orthogonal idempotents

$$T(n) \otimes V \twoheadrightarrow T(m) \longrightarrow T(n) \otimes V$$