TALK #2 - Tilting Modules Series

$\overline{\mathbb{k}} = \mathbb{k}$ $\quad$ char $\mathbb{k} = p$

$\mathcal{A}$: $\mathbb{k}$-linear abelian cat

Λ poset

$\Omega \subseteq \Lambda$ an ideal if $\lambda \in \Omega \Rightarrow \mu \in \Omega \ \forall \mu \leq \lambda$

~~ass~~ Ω: an coideal if $\Lambda \setminus \Omega$: ideal

assume: simples in $\mathcal{A}$: $\mathbb{L}(\lambda)$ $\quad \lambda \in \Lambda$

Ω: ideal $\quad \mathcal{A}_{\Omega}$ the Serre subcat gen. by $\mathbb{L}(\lambda)$, $\lambda \in \Omega$

if Ω: coideal

$$\mathcal{A}^{\Omega} := \mathcal{A} / \mathcal{A}_{\Lambda \setminus \Omega}$$

Here $\mathcal{A}/\mathcal{B}$: the Serre quotient

obj: same as $\mathcal{A}$

mor: $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y) = \varinjlim \mathrm{Hom}_{\mathcal{A}}(X', Y/Y')$

$X' \subset X \quad X/X' \in \mathcal{B} \quad Y' \subseteq Y \quad Y' \in \mathcal{B}$

$\mathcal{A}$: Highest wt cat if

① $\{\mu \mid \mu \leq \lambda\}$ is finite $\forall \lambda \in \Lambda$

② $\mathrm{Hom}(\mathbb{L}(\lambda), \mathbb{L}(\lambda)) = \Bbbk$

③ $\exists$ obj $\nabla(\lambda), \Delta(\lambda)$ $\forall \lambda \in \Lambda$

standard obj

$\Delta(\lambda) \twoheadrightarrow \mathbb{L}(\lambda)$ projective cover in $\mathcal{A}_{\leq \lambda}$

kernel in $\mathcal{A}_{< \lambda}$

costandard

$\mathbb{L}(\lambda) \hookrightarrow \nabla(\lambda)$ injective hull in $\mathcal{A}_{\leq \lambda}$

cokernel in $\mathcal{A}_{<\lambda}$

④ $\mathrm{Ext}^2(\Delta(\lambda), \nabla(\mu)) = 0$ $\forall \lambda, \mu$

An object is <u>tilting</u> if it admits both a standard filtration & a costandard filtration

if $\mathcal{A}$: Krull-Schmidt

tiltings: $\mathbb{T}(\lambda)$ $\lambda \in \Lambda$

and $(\mathbb{T}(\lambda), \nabla(\mu)) \neq 0 \Rightarrow \mu \leq \lambda$

and $(\mathbb{T}(\lambda), \nabla(\lambda)) = 1$

<u>Remark</u> in $\mathcal{A}^{\geq \lambda}$, $\mathbb{T}(\lambda), \mathbb{L}(\lambda), \nabla(\lambda), \Delta(\lambda)$ coincide

G: reductive algebraic gp

T, $W_f = N_G(T)$ $\quad X = X(T)$ $\quad X^+ \subset X$ dominant wts

Φ, set of roots $\quad \Phi^+$: simple roo.

$\mathrm{Rep}(G)$:

$\sim$ $\quad M \sim N \iff \mathrm{Ext}^1(M,N) \neq 0$

blocks: equivalent classes

$$W = \underset{w}{W_f} \ltimes \underset{t_\lambda^\vee}{\mathbb{Z}\Phi} \curvearrowright X$$

$$w t_\lambda \bullet \mu = w(\mu + p\lambda + \rho) - \rho$$

The linkage principle:

$$\mathbb{L}(\lambda) \sim \mathbb{L}(\mu) \Rightarrow \lambda = w \bullet \mu \quad \exists\, w \in W$$

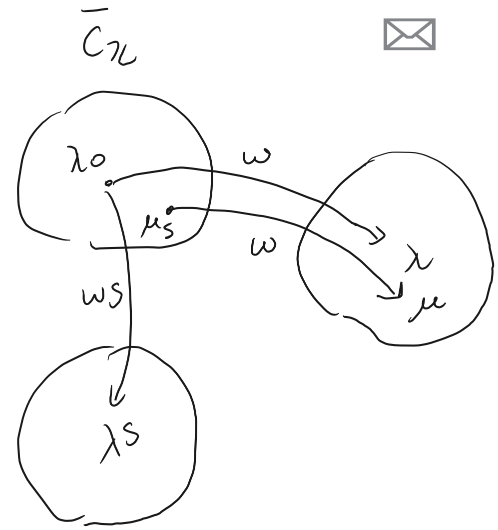
alcoves: connected component of

$$X \otimes_{\mathbb{Z}} \mathbb{R} \setminus \bigcup_{n \in \mathbb{Z}} \bigcup_{\alpha \in \Phi^+} \{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \lambda + \rho, \alpha^\vee \rangle = pn\}$$

W acts on the alcoves simply transitively

~~fund~~ $\quad C = \{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R} \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle < p \quad \forall \alpha \in \Phi^+\}$

$C_{\mathbb{Z}}$ $\quad \ldots \lambda \in X \ldots\ldots$



Choose $\lambda_0 \in C_{\mathbb{Z}}$

and $\lambda \in W \bullet \lambda_0 \cap X^+$

then $\lambda = w \bullet \lambda_0$ for some w

$$\lambda^s = ws \bullet \lambda_0$$

~~Choose~~ $\cancel{\mu_0}$

~~let~~ s : simple reflection

i.e. whose reflection hyperplane meets $\overline{C}$ in a codim 1 facet

choose $\mu_s \in \overline{C}_{\mathbb{Z}}$ contained in this hyperplane, but ~~does~~ is not contained in any ~~other~~ hyperplans

let $\mathrm{Rep}_0(G)$: the Serre subcat containing $\cancel{\mathbb{L}(\lambda)}$ $\mathbb{L}(\gamma)$ $\forall \gamma \in W \circ \lambda_0 \cap X^+$

$\mathrm{Rep}_s(G)$: ...... $\mathbb{L}(\gamma)$ $\forall \gamma \in W \circ \mu_s \cap X^+$

$\mathrm{pr}_\gamma(V)$ = the direct summand in V belonging to the block containing $\mathbb{L}(\gamma)$

$$T^s = T_{\lambda_0}^{\mu_s} \ \mathrm{Rep}_0(G) \to \mathrm{Rep}_s(G)$$
$$V \mapsto \mathrm{pr}_{\mu_s}(\mathbb{L}(\nu_1) \otimes \mathrm{pr}_{\lambda_0}(V))$$

Here, $\nu_1 \in W_f(\mu_s - \lambda_0)$

Fact

① $T_{\lambda_0}^{\mu_s}$ is exact

② $T_{\lambda_0}^{\mu_s}$, $T_{\mu_s}^{\lambda_0}$ adjoint to each other

③ if $\lambda \uparrow \lambda^s$

$$T_s(\mathcal{T}(\mu)) \cong \mathcal{T}(\lambda^s)$$

$$T^s(\mathcal{T}(\lambda)) \cong \mathcal{T}(\mu) \oplus \mathcal{T}(\mu)$$

i.e. $\mathcal{T}(\mu) \hookrightarrow T^s(\mathcal{T}(\lambda)) \twoheadrightarrow \mathcal{T}(\mu)$

Still assume $\mathcal{A}$: highest wt cat

if M: has a costandard filtration

A canonical ∇-flag of M is a $\{\Gamma_\Omega M\}$

$\forall\, \Omega \subseteq \Lambda$ ideal s.t.

① $\bigcup_\Omega \Gamma_\Omega M = M$, $\bigcap_\Omega \Gamma_\Omega M = 0$

② if $\Omega \subseteq \Omega'$, $\Gamma_\Omega M \subset \Gamma_{\Omega'} M$

③ $\lambda \in \Omega$ maximal

$\Gamma_\Omega M / \Gamma_{\Omega \setminus \{\lambda\}} M$ is a direct sum of $\nabla(\lambda)$'s

fixed

- $\mathbb{T}$ exists and unique
- behaves well after restr. / quot.

A section of the canonical ∇-flag of M

is $(\Pi, e, \{\varphi_\pi^M\}_{\pi \in \Pi})$

s.t. $\quad e : \Pi \longrightarrow \Lambda$

$$\varphi_\pi^M : \mathbb{T}(e(\pi)) \longrightarrow M$$

and $\quad \{\varphi_\pi^M\}$ for $\pi \in e^{-1}(\lambda) \quad \lambda \in \Lambda$

form a basis in $\quad \operatorname{Hom}_{\mathcal{A}^{\geq\lambda}}(\mathbb{T}(\lambda), M)$

- behaves well after restr. / quot.

$T^s(\Pi, e, \varphi_\pi^M)$

$(\Pi', e', \varphi_\pi^{T^sM})$:

$\Pi' = \{\pi \in \Pi \mid e(\pi)^s \in X^+\}$

$\pi \in \Pi'$. $\quad e'(\pi) =$ the unique wt μ in the closure of the alcove containing λ (in $X_s^+ = W \cdot \mu_s \cap X^+$)

when $\lambda^s \uparrow \lambda$:

$$\varphi_{\pi}^{T^s M} : \quad \mp(\mu) \to T^s T_s \mp(\mu) \xrightarrow{\sim} T^s \mp(\lambda) \to T^s M$$

when $\lambda \uparrow \lambda^s$,

$$\varphi_{\pi}^{T^s M} \quad \mp(\mu) \hookrightarrow T^s \mp(\lambda) \longrightarrow T^s M$$

Th (RW) $\quad (\Pi', e', \varphi_{\pi}^{T^s M})$ is a section on $T^s M$

$$T_s (\Pi, e, \varphi_{\pi}^{M})$$

$\Pi' = \Pi \times \{0,1\}$

$e'(\pi, 0)$ and $e'(\pi, 1)$ defined s.t.

- $e'(\pi, 0)^s = e'(\pi, 1)$
- is in the closure of the alcove containing $\mu = e(\pi)$ $\quad e'(\pi, 0)$ or $e'(\pi, 1)$
- $e'(\pi, 0) \uparrow e'(\pi, 1)$

$$\varphi^{T_s M}_{(\pi,0)} : \quad \mathcal{T}^{s}\,\mathcal{T}(e'(\pi,0)) \longrightarrow T_s T^{s}\,\mathcal{T}(e'(\pi,0))$$

$$\longrightarrow T_s\,\mathcal{T}(\mu) \longrightarrow T_s M$$

$$\varphi^{T_s M}_{(\pi,1)} : \quad \mathcal{T}(e'(\pi,1)) \simeq T_s\,\mathcal{T}(\mu) \to T_s M$$

(RW) · This is a Section !

$$\Theta_s = T_s T^s : \mathrm{Rep}_0(G) \to \mathrm{Rep}_0(G)$$

$\underline{w}$ reduced exp of w

$\underline{w} = (s_{i_1}, \dots, s_{i_t})$ $\quad s_{i_j} \in W$ simple refl.

$$\Theta(\underline{w}) = \Theta_{s_{i_1}} \circ \cdots \circ \Theta_{s_{i_t}}$$

$$\mathcal{T}(\underline{w}) = \Theta(\underline{w})(\mathcal{T}(\lambda_0))$$