THE SL_2 TILTING CATEGORY

in the mixed case

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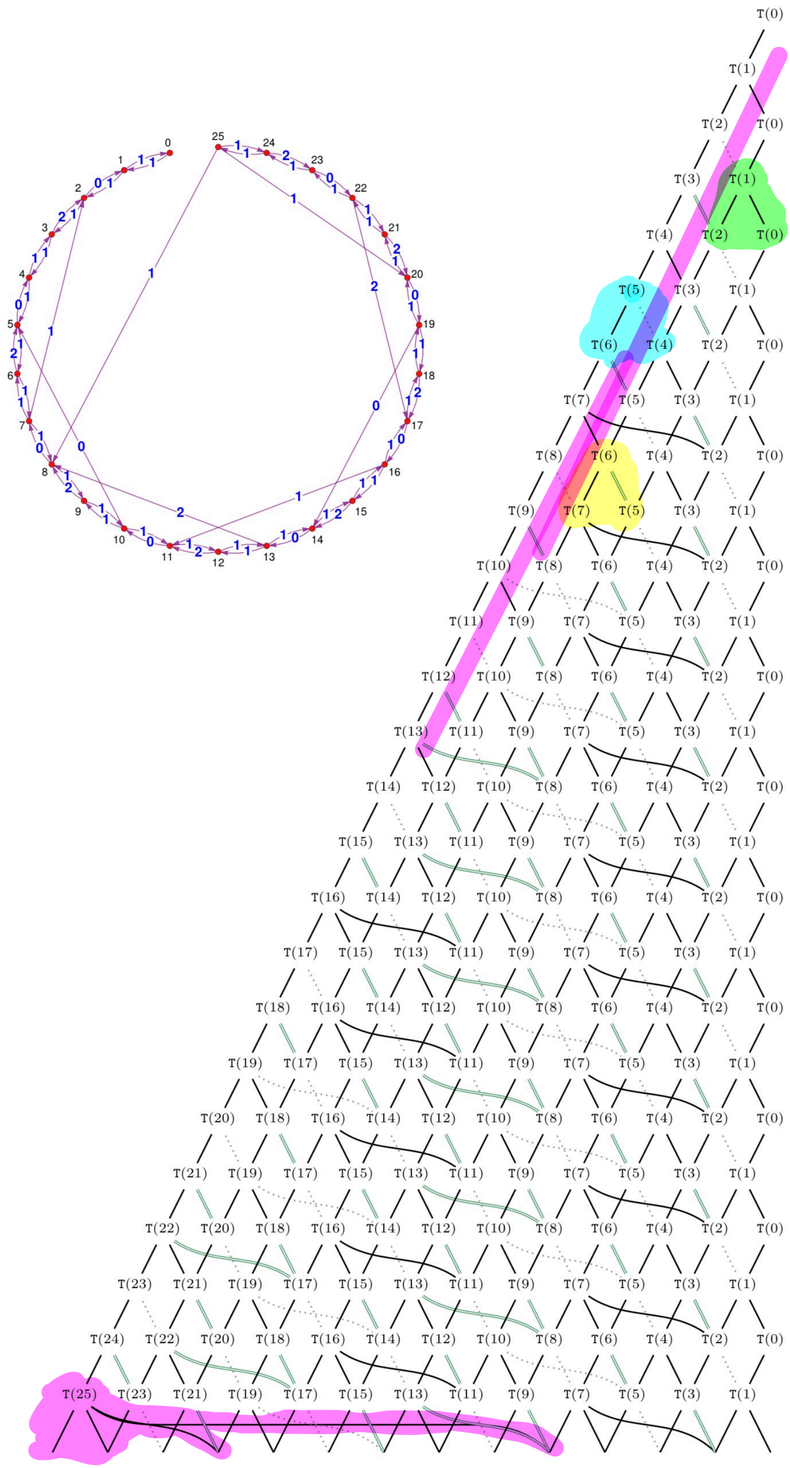


$-\otimes V$

$p=3$

FIGURE 3. *Top left picture:* The full subgraph of the fusion graph for T(1) in

p-Jones Wenzl idempotents

e.g. $p=3$ $n=16$

$n+1 = [1\ 2\ 2]_3$

$\mathrm{supp}(n+1) = \{[1\ 2\ 2]_3,\ [1\ 2\ {-2}]_3,\ [1\ {-2},\ 2]_3,\ [1,\ {-2},\ {-2}]_3\}$

$= \{17, 13, 5, 1\}$

$T(16)$:

$\Delta(16)$

|

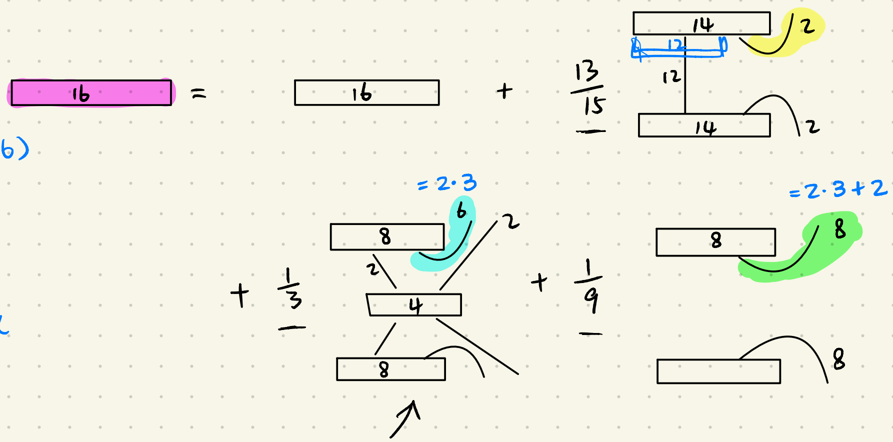
$T(16) \to \Delta(12) \to T(16)$

|

$\Delta(4)$

|

$\Delta(0) = \mathbb{k}$



$T(n) \twoheadrightarrow \Delta(m) \longrightarrow T(n)$ $\quad m \leq n$

(p, ℓ) - Jones Wenzl idempotents

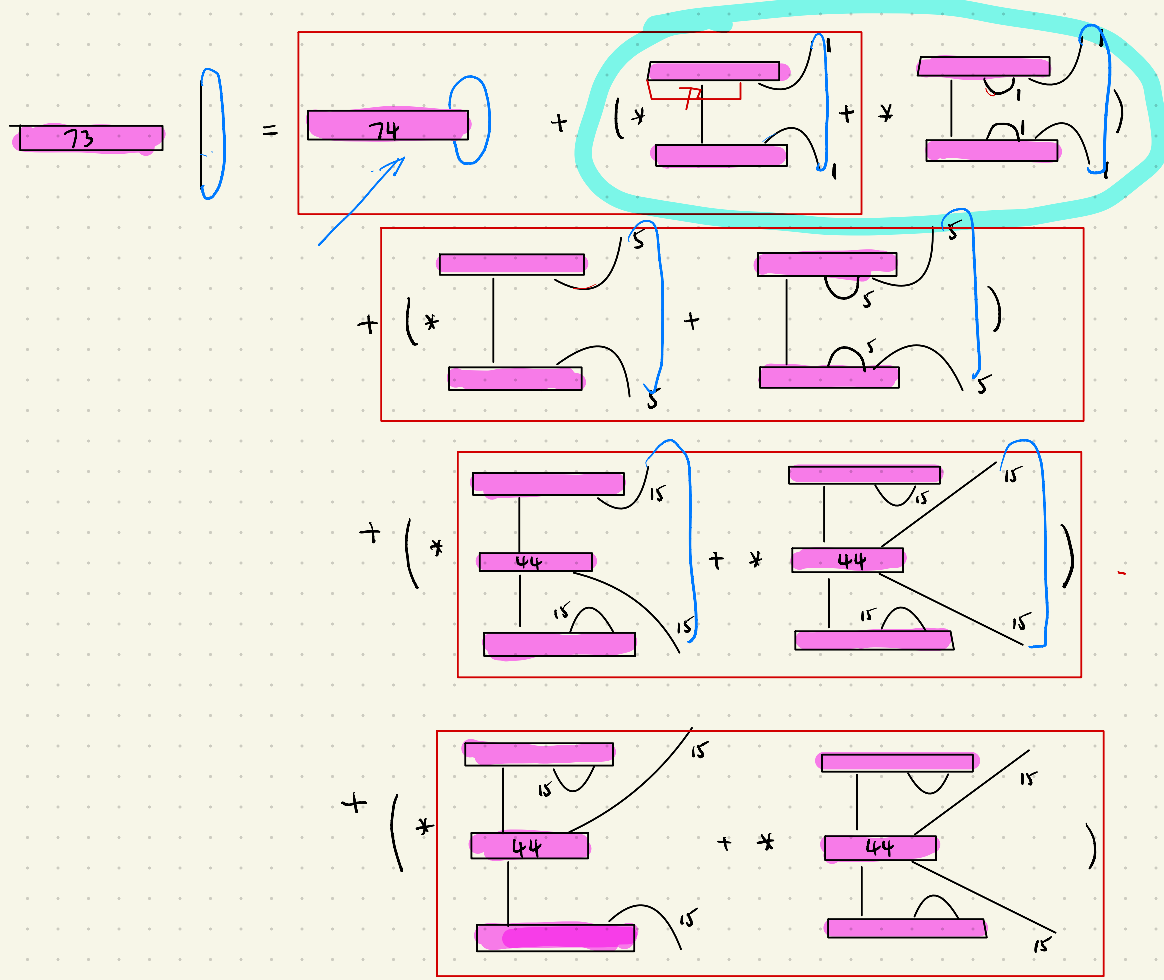
$p=3,\ \ell=5$ (p, ℓ)-adic exp $n = \sum a_i p^{(i)}$ $p^{(i)} = p^{i-1}\ell$

e.g. $n = 73$

$$n+1 = [1\ 1\ 2\ 4]_{3,5} = 1\cdot\underbrace{(3^2\cdot 5)} + 1\cdot\underbrace{(3\cdot 5)} + 2\cdot 5 + 4 = 74$$

"long edges" "double edge"

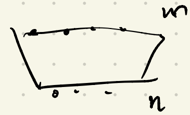
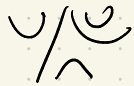
$$T(73) \otimes T(1) \simeq T(74) \oplus T(72) \oplus T(64) \oplus T(46)^{\oplus 2}$$

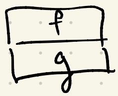
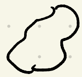


(Jones-Wenzl idempotents in the mixed case)

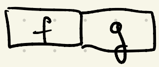
$TL(\delta)$: Temperley-Lieb cat. $\Bbbk$. alg closed

obj: $n \in \mathbb{Z}_{\geq 0}$

mor: $Hom(n,m) = \Bbbk$-span   $\in Hom(3,7)$

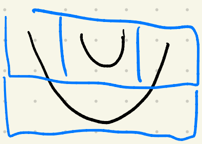
Composition.  $= fg$  $= \delta$

monoidal: $n \otimes m = n+m$

 $= f \otimes g$

$\mathcal{F}$: $TL(\delta) \longrightarrow Sl_2\text{-mod}$

$n \longmapsto V^{\otimes n}$ $V = \Bbbk^2$

 $(1 \otimes \cup \otimes 1) \circ \cup$

 $\longmapsto$ coev: $\Bbbk \to V \otimes V \overset{\sim}{=} V \otimes V^*$

$1 \mapsto v_1 \otimes v_1^* \otimes v_2 \otimes v_2^*$

$\overset{*}{V}$ is V^* (IS)

 $\longmapsto$ ev: $V^* \otimes V \to \Bbbk$

$f \otimes v = f(v)$

 $\longmapsto$ id: $V \to V$

$\underset{IS}{V}$

(Rumer-Teller-Weyl) $\mathcal{F}$ is full on the subcat generated by V, V^*

i.e. $Hom(V^{\otimes m}, V^{\otimes n})$ can be rep. as TL diagrams.

$SL_2(\mathbb{k})$ char $\mathbb{k}=0$

$V = L(\varepsilon_1)$ ε_1 fundamental weight

$$\begin{array}{c} \uparrow \\ L(n\varepsilon_1) = L(n) \quad n \in \mathbb{Z}_{\geq 0} \end{array}$$

Weyl modules: $S^n(V) = \Delta(n)$ $n \in \mathbb{Z}_{\geq 0}$

$$\Delta(n) \twoheadrightarrow \underset{\text{head}}{L(n)}$$

Dual Weyl modules: $\nabla(n) := \Delta(n)^*$

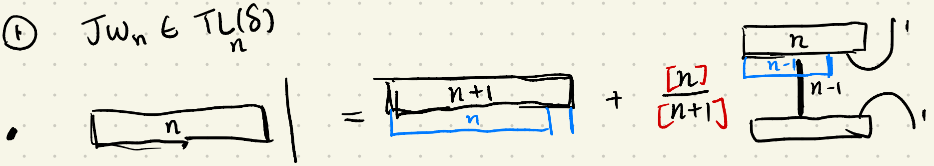
Tilting. $T(n) = \Delta(n)$

$$\underset{\substack{\| \\ V}}{\Delta(n) \otimes \Delta(1)} = \underbrace{\Delta(n+1)} \oplus \Delta(n-1)$$

$$\Delta(n) \subseteq V^{\otimes n}$$

$$JW_n : \left(V^{\otimes(n)} \twoheadrightarrow \underset{\substack{\| \\ T(n)}}{\Delta(n)} \hookrightarrow V^{\otimes(n)} \right) \in TL^{(\delta)}(n,n)$$

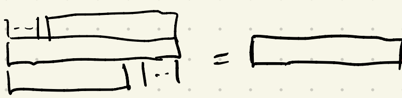
① $JW_n \in TL_n(\delta)$

• 

$\delta = 2$

quantum $\delta = [2]_q$

② 1) uncappable 

2) absorption 

char $\mathbb{k}=0$

char $\mathbb{k} = p$ "q"

$$V^{\otimes(n+1)} \to \underbrace{\Delta(n)} \otimes V \hookrightarrow V^{\otimes(n+1)}$$

$$V^{\otimes(n+1)} \to \Delta(n+1) \to V^{\otimes(n+1)}$$

$$\to \Delta(n-1) \to$$

char $\Bbbk = p$

$\underline{\Delta(n)}$, $\nabla(n)$

Tilting $T(n)$: unique SL_2-mod w/ both a Δ-filtration (standard)

and a ∇-filtration (costandard)

- $\mathrm{Hom}(\Delta(n), T(n)) = \Bbbk$. $\mathrm{Hom}(T(n), \nabla(n)) = \Bbbk$

(Highest weight category Cline - Parshall - Scott
Dlab - Ringel)

$T(n)$: nontrivial Δ-flag

p - Jones Wenzls
(Burrull - Sentinelli - Libedinsky)

- orthogonal decomp.
- idempotents

$\oplus, \otimes$
Sub. cat gen. by tilting

$T(n) \otimes T(m) \simeq \oplus T(i)$

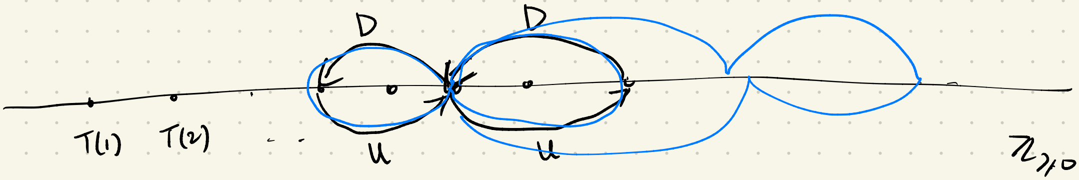
(Tubbenhauer - Wedrich)

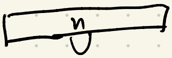
$$\mathcal{F} \quad \bigoplus_{\substack{n,m \\ \in \mathbb{Z}_{\geq 0}}} e_n TL\, e_m \xrightarrow{\sim} \mathrm{Tilt}(SL_2\text{-mod})$$

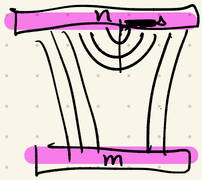
$$e_n = \boxed{\ n\ } \qquad e_m \longmapsto T(m)$$

This cat has a quiver rep:

$$\bigoplus_{i,j \in \mathbb{Z}_{\geq 0}} \mathrm{Hom}(T(i), T(j)) \simeq \text{Path alg } (Z)$$







unique

$\mathrm{Hom}(\Delta(n), \Delta(m)) = 0$

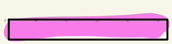
- Complete list of relations

$$\begin{cases} DU = UD + UUDD + \text{lowers} \ldots \\ DD = 0 \quad UU = 0 \\ \vdots \end{cases}$$

- (~~Bo~~ Sutton - T - W - Z.)
  - everything holds in (p, ℓ)-mixed case
  - char $\mathbb{k} = p$
  - ℓ: smallest integer. $q^{2\ell} = 1$ $\quad q \in \mathbb{k}$
  - (p, ℓ)-JW w.d.

$$n \longrightarrow [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

- Monoidal structure

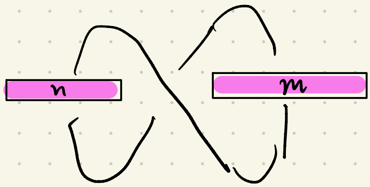
Formula for  | $T(n) \otimes V$

Future work

- link invariants.

S-matrix

$S_{n,m}$



$:= \beta_{V,W} : V \otimes W \to W \otimes V$

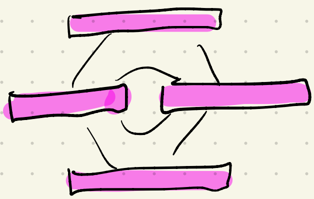
R-matrix

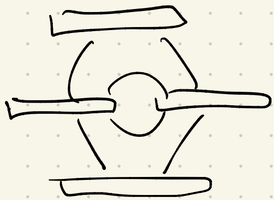
$q \rightsquigarrow 1$

$= q\,)(+ q^{-1}$

$= q^{-1})(+ q$

- " $S_{n,m}$ invertible $\Longleftrightarrow$ category is modular)"





θ network

$T(n) \otimes T(m)$

$\nearrow \quad \nearrow$

$\Delta(n) = [n]_q$