$K(\infty)$: Hopf $\mathbb{Z}$-alg

$\circ, \Delta$

$\iota : \mathbb{Z} \to K(\infty)$

$1 \mapsto \mathbb{1}$. $\mathcal{H}_0$-mod

$\varepsilon : \underbrace{K(\infty)} \to \mathbb{Z}$

proj to $K_0(\mathrm{Rep}_I \mathcal{H}_0)$

Th 9.2.1 $\quad (K(\infty), \circ, \Delta, \iota, \varepsilon)$ is Hopf alg

$\Delta : K(\infty) \to K(\infty) \otimes K(\infty)$: alg homo regarding $\circ$.

$$\Delta([M] \circ [N]) = \underbrace{\Delta([M])}_{\in \underline{K(\infty)} \otimes K(\infty)} \circ \underbrace{\Delta([N])} \leftarrow$$

<u>Proof</u> Using Mackey Theorem

$\mathrm{Res}^n_\mu \mathrm{Ind}^n_\nu M$ has composition factors

$$\sum_{x \in D_{\mu,\nu}} \underbrace{\mathrm{Ind}^{\mu}_{\mu \cap x\nu} {}^x\left(\mathrm{Res}^{\nu}_{x^{-1}\mu \cap \nu} M\right)}$$

E.g. $\quad M$: $\mathcal{H}_2$-mod $\qquad N$: $\mathcal{H}_2$-mod

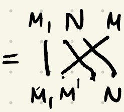
$$[M] \circ [N] = [\mathrm{Ind}^4_{2,2} M \otimes N]$$

$$\Delta([\mathrm{Ind}^4_{2,2} M \otimes N]) = [\mathrm{Res}^4_{3,1} \mathrm{Ind}^4_{2,2} M \otimes N] + [\mathrm{Res}^4_{2,2} \cdots] + [\mathrm{Res}^4_{4,0} \cdots]$$

$$\left[\operatorname{Res}^{4}_{3,1}\operatorname{Ind}^{4}_{2,2} M\otimes N\right] = \operatorname{Ind}^{3,1}_{\underline{\mu\cap x\nu}}\operatorname{Res}^{2,2}_{x^{-1}\mu\cap\nu} M\otimes N$$

$$x^{-1}S_\mu x \cap S_\nu$$

$H_1\otimes H_2\otimes H_1$

$x =$ 

$$= \operatorname{Ind}^{3,1}_{1,2,1}{}^{x}\left(\operatorname{Res}^{2,2}_{1,1,2}\underbrace{M\otimes N}_{H_2 \quad H_2}\right)$$

$$+ \; I \ldots\ldots$$

$H_1\otimes H_1$

$$\Delta([M]) = \underbrace{[M_0]\otimes[M^2]}_{\cancel{H_0}\otimes H_2,\ \Bbbk\otimes M} + \underbrace{[M_1]\otimes[M']}_{H_1\otimes H_1} + \underbrace{[M_2]\otimes[M^0]}_{H_2\otimes H_0,\ M\otimes \Bbbk} \leftarrow$$

$$= \operatorname{Ind}^{3,1}_{1,2,1}{}^{x}\left(M_1\otimes M'\otimes N\right)$$

$$= \operatorname{Ind}^{3,1}_{1,2,1}\left(M_1\otimes N\otimes M'\right)$$

$$\Delta([M]\circ[N]) = \left[\operatorname{Ind}^{3}_{1,2} M_1\otimes N\right]\otimes[M'] \leftarrow$$

$$+ \cdots$$

$$\underline{\Delta([M])}\circ\underline{\Delta([N])}$$

$$\left([M_1]\otimes[M']\right)\cdot\left([N_2]\otimes[N^0]\right)$$

$$= \left([M_1]\cdot[N_2]\right)\otimes\left([M']\cdot[N^0]\right)$$

$$= \left[\operatorname{ind}^{3}_{1,2} M_1\otimes N_2\right]\otimes\left[\operatorname{ind}^{1}_{1,0} M'\otimes N^0\right] \leftarrow$$

Lemma 9.2.4 $\quad e_i : K(\infty)\to K(\infty)$

e_i's satisfy the Serre relations.

① $e_ie_j = e_je_i \qquad |i-j|>1$ ✓

② $\underline{e_i^2}e_j + e_je_i^2 = 2e_ie_je_i \qquad p>2$

$e_i^3e_j + 3e_ie_je_i^2 = 3e_i^2e_je_i + e_je_i^3 \qquad p=2$

$e_i^r \cancel{=} r!\, e_i^{(r)}$

$e_i^r[M] \cancel{=} r!\, e_i^{(r)}[M]$

$e_i^r = (r!)\, e_i^{(r)}$ as operators on $K(\infty)$

$$\frac{e_i}{r!} \cdot \frac{e_i}{r!} \cdot \frac{e_j}{r!} + \frac{e_j}{r!} \cdot \frac{e_i}{r!} \cdot \frac{e_i}{r!} = \cdots$$

$$e_i^{(r)} e_i^{(r)} e_j^{(r)} + e_j^{(r)}$$

e I

Proof.
$$\Delta^n_{n-1,1}([M]) = \sum_M \underbrace{[M_{(1)}]}_{\mathcal{H}_{n-1}} \otimes \underbrace{[M_{(2)}]}_{\mathcal{H}_1}$$

$$\underset{\llcorner\!\lrcorner}{e_i}([M]) = \sum_{M_{(2)} \simeq \underbrace{L(i)}} [M_{(1)}] \qquad L(i) \curvearrowleft x_n \text{ acts by } i$$

operator
$$e_i(W) = \operatorname{Res}^{n-1,1}_{n-1} W_i$$
$\uparrow$ generalized i-eigenspace under the action of x_n

$$\left(\Delta^{n-1}_{n-2,1} \otimes \mathrm{id}\right)\left(\Delta^n_{n-1,1}([M])\right) = \underbrace{\sum \left(\Delta^{n-1}_{\underbrace{n-2,1}} [M_{(1)}]\right) \otimes [M_{(2)}]}$$

$$= \sum_M M_{(1)} \otimes M_{(2)} \otimes M_{(3)}$$

$$= \left(\mathrm{id} \otimes \Delta^2_{1,1}\right)\left(\Delta^n_{n-2,2}([M])\right)$$

$$e_i^2 e_j + e_j e_i^2 = 2 e_i e_j e_i \qquad (*)$$

$$\Delta^n_{n-3,1,1,1}(M) = \left(\mathrm{id} \otimes \Delta^3_{1,1,1}\right)\left(\Delta^n_{n-3,3}(M)\right)$$
$$= \left(\mathrm{id} \otimes \Delta^3_{1,1,1}\right)\left(\sum_i M_i \otimes \underbrace{N_i}\right)$$

sufficient to look at terms where $N_i \simeq L(i,i,j) \leftarrow$
or $L(i,j,i) \leftarrow$
or $L(j,i,i) \leftarrow$

o.w. $(*)$ acts as zero

$$\left(\mathrm{id} \otimes \Delta^3_{1,1,1}\right)\left(N \otimes L(i,j,i)\right) = [N] \otimes \mathrm{ch}\, L(i,j,i)$$

(Lemma 6.2.2)

$$= [N] \otimes \left(\underbrace{[L(i) \otimes L(j) \otimes \underline{L(i)}]}_{} + 2 [L(i) \otimes L(i) \otimes L(j)] \right)$$

$$e_i^2 e_j ([M]) = e_i^2 \left([N] \otimes 2 [L(i) \otimes L(i)] \right)$$

$$= e_i \left([N] \otimes 2 [L(i)] \right) = 2[N]$$

$$\underline{e_j} e_i^2 ([M]) = 0$$

j i i

$$2 e_i e_j e_i ([M]) = 2[N]$$

$L(i) \otimes L(j) \otimes L(i)$

$$\Delta : \underbrace{K(\infty) \to K(\infty) \otimes K(\infty)}$$

$$\Delta^\lambda : \underline{K(\lambda)} \to \underline{K(\infty)} \otimes K(\infty)$$

$\mathcal{H}_n$ $\quad$ $\mathcal{H}_{n_1} \otimes \mathcal{H}_{n_2}$ $\quad$ $n_1 + n_2 = n$

x_1^4 $\quad$ x_1^4

$K(\lambda) \otimes K(\infty)$

$K(\infty)^*$: restricted dual

$= \{ f : K(\infty) \to \mathbb{Z} \mid f :$ zero on all but finitely many elts in $\underline{B(\infty)}$ $\}$

$B(\infty)$: $\mathbb{Z}$-basis for $K(\infty)$

$\underline{K(\infty)^* \curvearrowright K(\lambda)}$

$f \in K(\infty)^*$

$$\overset{\infty}{K(\lambda)} \xrightarrow{\Delta^\lambda} \overset{\infty}{K(\lambda) \otimes K(\infty)} \xrightarrow{id \otimes f} \overset{\infty}{K(\lambda) \otimes \mathbb{Z}} \simeq \underline{K(\lambda)}$$

action of f : composition

$\underline{K(\infty)^* \curvearrowright K(\infty)}$ $\underline{\text{faithful}}$

$M \in B(\infty)$, $\delta_M \in K(\infty)^*$

Lemma 9.25. $e_i^{(r)} : K(\infty) \to K(\infty)$ i.e. $\curvearrowright K(\infty)$

$\underbrace{e_i^{(r)}}$ agrees with $\underbrace{\delta_{L(i^r)}} \in K(\infty)^*$

$(r!)\, e_i^{(r)} \simeq e_i$

Lemma 9.26 $\quad \pi : \underbrace{U_{\mathbb{Z}}^+} \to K(\infty)^*$.

upper half of $U(sl_n)$

i.e. gen. by $\{E_i\} / \sim$ serre rel

$$\underbrace{E_i^{(r)}}_{= \frac{E_i^r}{r!}} = \frac{1}{r!} \underbrace{E_i^r} \mapsto \underbrace{\delta_{L(i^r)}}_{= e_i^{(r)}}$$

① coalg hom.

$E_i \mapsto \delta_{L(i)}$

$\frac{1}{r!} \delta_{L(i)}^r = \delta_{L(i^r)}$

$$K(\infty)^* \to K(\infty)^* \otimes K(\infty)^*$$

$$f \mapsto \underbrace{\sum_i f_i \otimes g_i}$$

$$K(\infty) \otimes K(\infty) \xrightarrow{\circ} K(\infty) \xrightarrow{f} \mathbb{Z}$$

$[M] \otimes [N]$

s.t. $\sum_i f_i \otimes g_i \Big([M] \otimes [N] \Big) = f_i([M]) \cdot g_i([N])$

$$= f([M] \circ [N])$$

$$K(\infty)^* \otimes K(\infty)^* = \big(K(\infty) \otimes K(\infty) \big)^*$$

$$\Delta(E_i^{(r)}) = E_i^{(r)} \otimes 1 + 1 \otimes E_i^{(r)}$$

$$\Delta(\delta_{L(i^r)}) = \delta_{L(i^r)} \otimes 1 + 1 \otimes \delta_{L(i^r)}$$

② alg hom

$\delta_{L(i^r)}$'s satisfy Serre Relations as operators on $\underbrace{K(\infty)}$

$\delta_{L(i^r)}$'s - - - - - - - - - in $K(\infty)^*$

(as an algebra)