Kleshchev §13

§13.1 Twisted Symmetric Group

T_n : superalg gen. by $t_1, \dots, t_{n-1}$ subject to

$$t_i^2 = 1$$

$$t_i t_j = - t_j t_i \qquad |i-j|>1$$

$$t_i t_j t_i = t_j t_i t_j \qquad |i-j| = 1$$

$t_{i_1} \cdots t_{i_r}$ even (odd) $\iff$ r even (odd)

$$g = s_{i_1} \cdots s_{i_r}$$

$$t_g = t_{i_1} \cdots t_{i_r}$$

$$(T_n)_{\bar 0} = \mathrm{Span}\{t_g \mid g \in A_n\}$$

$$(T_n)_{\bar 1} = \mathrm{Span}\{t_g \mid g \in S_n / A_n\}$$

$\tau_1 : T_n \to T_n \qquad t_i \mapsto -t_i$

$1 \le i < j \le n-1$

odd

$$[i\ j] = (-1)^{i-j-1}\, t_j \cdots t_{i+1} t_i t_{i+1} \cdots t_j$$

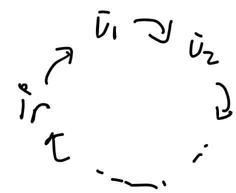
$$(i\ j)$$

FACT $\quad [i\ j]^2 = 1$

$$[i\ j][l\ k] = -[l\ k][i\ j] \qquad \{i, j\} \cap \{l, k\} = \phi$$

$$[i\ j][j\ k][i\ j] = [j\ k][i\ j][j\ k] = [k\ i]$$

$$[i_1\ i_2 \ldots\ i_r] = [i_{r-1}\ i_r] \cdots [i_1\ i_r]$$



$$M_k = \sum_{i=1}^{k-1} [i\ k] \qquad \text{odd}$$

$$M_1 = 0$$

Fact

$$t_i M_k = \begin{cases} -M_k t_i & i \neq k, k-1 \\ -M_{k-1} t_i + 1 & i = k-1 \\ -M_{k+1} t_i + 1 & i = k \end{cases}$$

t_i Commutes with $M_k^2 \quad i \neq k, k-1$

$$M_i^2 + M_{i+1}^2, \quad M_i^2 M_{i+1}^2$$

<u>Remark</u> Symmetric poly in $M_i^2 \subseteq$ Center of T_n

§ 13.2 Sergeev Alg.

$\mathcal{C}_n$: Clifford alg

$\mathcal{Y}_n := \mathcal{T}_n \otimes \mathcal{C}_n$

$\tau_2 : \mathcal{C}_n \to \mathcal{C}_n \quad c_i \mapsto c_i$ anti~~morphism~~ involution

$\mathcal{Y}_n$: $t_i \otimes 1 =: t_i$

$1 \otimes c_i =: c_i$

$\tau_1 \otimes \tau_2 : \mathcal{Y}_n \to \mathcal{Y}_n$ anti-involution

~~$n = 2k$~~ $\quad \mathcal{C}_n \simeq \mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_1$ $\qquad$ unique irred mod

$n = 2k \qquad \mathcal{C}_n \simeq Mat_{2^{k-1}, 2^{k-1}} \qquad U_n \quad$ type M

$n = 2k-1 \qquad \mathcal{C}_n \simeq Q_{2^{k-1}} \qquad U_n \quad$ type Q

$\mathbb{I}_n : \mathcal{T}_n\text{-smod} \to \mathcal{Y}_n\text{-smod}$

$V \mapsto V \boxtimes U_n$

$\mathbb{R}_n : \mathcal{Y}_n\text{-smod} \to \mathcal{T}_n\text{-smod}$

$W \mapsto \mathrm{Hom}_{\mathcal{C}_n}(U_n, W)$

$\mathrm{Res}^{\mathcal{Y}_n}_{\mathcal{Y}_{n-1}} \qquad \mathrm{Res}^{\mathcal{T}_n}_{\mathcal{T}_{n-1}}$

$\mathrm{Ind}^{\mathcal{Y}_n}_{\mathcal{Y}_{n-1}} \qquad \mathrm{Ind}^{\mathcal{T}_n}_{\mathcal{T}_{n-1}}$

Proposition. ① n even. $\mathbb{R}_n, \mathbb{I}_n$ biadjoint equivalence of categories. Type preserving

② n odd. biadjoint. $\mathbb{R}_n \circ \mathbb{I}_n \simeq \mathrm{id} \oplus \Pi$, $\quad \mathbb{I}_n \circ \mathbb{R}_n \simeq \mathrm{id} \oplus \Pi$

$\mathbb{I}_n$ ($\mathbb{R}_n$) induces a bijection

$$\left\{ \begin{array}{l} \text{irred } \widetilde{\mathcal{T}}_n\text{-smod } (\mathcal{Y}_n\text{-smod}) \\ \text{of type M} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{irred } \mathcal{Y}_n\text{-smod } (\widetilde{\mathcal{T}}_n\text{-smod}) \\ \text{of type Q} \end{array} \right\}$$

$$\mathbb{I}_n(Q) = M \oplus \Pi M$$

$$\begin{array}{ccc} \widetilde{\mathcal{T}}_n\text{-smod} & & \mathcal{Y}_n\text{-smod} \\ \text{Type M} & \xrightarrow[\simeq]{\mathbb{I}_n} & \text{Type Q} \\ & & \\ \text{Type Q} & \xleftarrow[\mathbb{R}_n]{} & \text{Type M} \end{array}$$

Proof. ① n: even

α_V: $\mathbb{I}_n \circ \mathbb{R}_n \to \mathrm{id}$

$$\mathrm{Hom}_{\mathcal{C}_n}(U_n, V) \otimes U_n \longrightarrow V$$

$$f \otimes u \mapsto f(u)$$

surjective: $\forall\, v_0 \in V \quad \mathcal{C}_n . v_0 \subseteq V$ irred $\mathcal{C}_n$-mod

$$f: U_n \xrightarrow{\sim} \mathcal{C}_n . v_0 \hookrightarrow V \qquad f(u) = v_0$$
$$u \mapsto v_0 \mapsto v_0$$

dimensions match:

$$\dim V = (\text{mult of } U_n \text{ in } V \text{ as a direct summand}) \cdot \dim U_n$$

$$= \dim \operatorname{Hom}_{\mathcal{C}_n}(U_n, V) \cdot \dim U_n$$

β_W: $\mathrm{id} \to \mathbb{R}_n \circ \mathbb{I}_n$

$$W \to \operatorname{Hom}_{\mathcal{C}_n}(U_n, W \otimes U_n)$$

$$w \mapsto (\theta_w : u \mapsto w \otimes u)$$

β_W: injective.

dimensions match

α_V, β_W. natural isom. $\Rightarrow$ $\mathbb{I}_n, \mathbb{R}_n$. equiv. of categories

Type preserving

Note: $\mathbb{I}_n, \mathbb{R}_n$ exact

$- \boxtimes U_n$ exact

$\operatorname{Hom}_{\mathcal{C}_n}(P, -)$ exact $\Leftrightarrow$ P projective.

2 n: odd

α_V : $\mathbb{I}_n \circ \mathbb{R}_n \to \mathrm{id} \oplus \Pi$

$$\operatorname{Hom}_{\mathcal{C}_n}(U_n, V) \otimes U_n \longrightarrow V \oplus \Pi V$$

$$\theta \otimes u \mapsto (\theta(u), (-1)^{\bar{\theta}} \theta(J(u)))$$

$J \in \operatorname{End}_{\mathcal{C}_n}(U_n)$ odd

α_V: surj. $\quad (\theta(u), (-1)^{\bar{\theta}} \theta(J(u))) \in \operatorname{Im}$

$$U_n \xrightarrow{J} U_n \xrightarrow{\theta} V$$

$$\theta \circ J \otimes J(u) \mapsto (\theta \circ J(J(u)), (-1)^{\overline{\theta \circ J}} \theta \circ J(J(J(u)))$$

$$= (\theta(u), -(-1)^{\bar{\theta}} \theta(J(u)))$$

$(\theta(u), 0)$, $(0, (-1)^{\bar{\theta}}\, \theta(J(u))$ $\in \mathrm{Im}$.

dimensions:

$$\dim \mathrm{Hom}_{\mathcal{C}_n}(U_n, V) = 2\cdot(\text{multiplicity of } U_n \text{ as a direct summand of } V)$$

β_V: $\quad \mathrm{id} \oplus \Pi \longrightarrow \mathbb{R}_n \circ \mathbb{I}_n$

$$W \oplus \Pi W \longrightarrow \mathrm{Hom}_{\mathcal{C}_n}(U_n, W \otimes U_n)$$

$$(w, w') \mapsto (\theta_{w,w'} : u \mapsto w \otimes u + (-1)^{\overline{w'}}\, w' \otimes J(u))$$

β_V: inj. dimensions match.

$$\mathbb{I}_n \circ \mathbb{R}_n \xrightarrow{\sim} \mathrm{id} \oplus \Pi \xrightarrow{\pi} \mathrm{id} \qquad \alpha: \text{left adjoint to } \beta$$

$$\mathrm{id} \hookrightarrow \mathrm{id} \oplus \Pi \xrightarrow{\sim} \mathbb{R}_n \circ \mathbb{I}_n$$

$$\mathbb{I}_n \circ \mathbb{R}_n(V) \simeq V \oplus \Pi V$$

$$\mathbb{R}_n \circ \mathbb{I}_n(W) \simeq W \oplus \Pi W$$

$\mathbb{I}_n$ induces: inj

$$\mathbb{I}_n(M_1) \simeq \mathbb{I}_n(M_2)$$

$$\mathbb{R}_n(\mathbb{I}_n(M_1)) \simeq \mathbb{R}_n(\mathbb{I}_n(M_2))$$

$$M_1 \oplus \Pi M_1 \simeq M_2 \oplus \Pi M_2$$

surj $\quad \mathbb{I}_n(\mathbb{R}_n(Q)) \simeq Q \oplus \Pi Q$

$$\mathbb{R}_n(Q) \simeq M_1 \oplus M_2$$

$$\mathbb{I}_n(M_1) \simeq Q \text{ or } \Pi Q$$

$\mathbb{R}_n$ induces: inj.

surj $\quad \mathbb{R}_n(\mathbb{I}_n(Q)) \simeq Q \oplus \Pi Q$

$$\mathbb{R}_n(M \oplus \Pi M) \simeq Q \oplus \Pi Q$$