Kleshchev §13

Recall $\mathbb{I}_n : \mathcal{T}_n\text{-smod} \to \mathcal{Y}_n\text{-smod}$

$$V \mapsto V \boxtimes U_n$$

$\mathbb{R}_n : \mathcal{Y}_n\text{-smod} \to \mathcal{T}_n\text{-smod}$

$$W \mapsto \mathrm{Hom}_{\mathcal{C}_n}(U_n, W)$$

Prop. $\mathbb{I}_n, \mathbb{R}_n$: biadjoint

$$\mathrm{id} \xhookrightarrow{\iota} \mathrm{id} \oplus \Pi \xrightarrow{\beta} \mathbb{R}_n \circ \mathbb{I}_n$$

$$\mathrm{id} \oplus \Pi \underset{\alpha^{-1}}{\overset{\sim}{\dashrightarrow}} \mathbb{I}_n \circ \mathbb{R}_n$$

$$\mathbb{I}_n \circ \mathbb{R}_n \underset{\sim}{\xrightarrow{\alpha}} \mathrm{id} \oplus \Pi \xrightarrow{\pi} \mathrm{id}$$

$$\mathbb{R}_n \circ \mathbb{I}_n \underset{\beta^{-1}}{\dashrightarrow} \mathrm{id} \oplus \Pi$$

$$\sqrt{2}\,\alpha^{-1} \circ \iota, \quad \sqrt{2}\,\pi \circ \beta^{-1}.$$

$\tau : \mathcal{Y}_n \to \mathcal{Y}_n$. $\quad \tau_A : \mathcal{T}_n \to \mathcal{T}_n$, $\quad \tau_B : \mathcal{C}_n \to \mathcal{C}_n$

$$\tau = \tau_A \otimes \tau_B$$

$\tau : \quad V \mapsto V^{\tau}$

$$\tau \circ \mathbb{I}_n \cong \mathbb{I}_n \circ \tau$$

$$(V \boxtimes U_n)^{\tau} \simeq V^{\tau_A} \boxtimes U_n^{\tau_B} \simeq V^{\tau_A} \boxtimes U_n$$

$$\tau \circ \mathbb{R}_n \circ \tau \simeq \mathbb{R}_n$$

~~left~~ right adjoint to $\mathbb{I}_n$

$$\mathrm{Hom}(V, (\mathbb{R}_n W^{\tau})^{\tau}) \simeq \mathrm{Hom}(\mathbb{R}_n W^{\tau}, V^{\tau})$$
$$\simeq \mathrm{Hom}(W^{\tau}, \mathbb{I}_n V^{\tau})$$
$$\simeq \mathrm{Hom}(\mathbb{I}_n V, W)$$

$\mathbb{I}_n$: left adjoint to $\tau \circ \mathbb{R}_n \circ \tau$

n : even

$\cancel{\mathrm{Res}^{n}_{n-1} \circ \mathrm{IR}_n \simeq \mathrm{IR}_{n-1} \circ \mathrm{Res}^{n}_{n-1}}$

$$\mathrm{Res}^{\mathcal{Y}_n}_{\mathcal{Y}_{n-1}} \circ \Pi_n \simeq \Pi_{n-1} \, \mathrm{Res}^{\mathcal{T}_n}_{\mathcal{T}_{n-1}}$$

n : odd

$$\boxed{\mathrm{Res}^{\mathcal{T}_n}_{\mathcal{T}_{n-1}} \, \mathrm{IR}_n \simeq \mathrm{IR}_{n-1} \, \mathrm{Res}^{\mathcal{Y}_n}_{\mathcal{Y}_{n-1}}}$$

Proof

$$\mathrm{Res}^{\mathcal{Y}_n}_{\mathcal{Y}_{n-1}} V \boxtimes U_n \simeq \mathrm{Res}^{\mathcal{T}_n}_{\mathcal{T}_{n-1}} V \boxtimes \mathrm{Res}^{\mathcal{C}_n}_{\mathcal{C}_{n-1}} U_n \simeq \mathrm{Res}^{\mathcal{T}_n}_{\mathcal{T}_{n-1}} V \boxtimes U_{n-1}$$

$n = 2k$ $\quad \mathcal{C}_n \simeq \mathrm{Mat}_{2^{k-1}, 2^{k-1}}$ $\quad$ sdim $(2^{k-1}, 2^{k-1})$

$n = 2k-1$ $\quad \mathcal{C}_{n-1} \simeq Q_{2^{k-1}, 2^{k-1}}$

n = odd :

$$\mathrm{Hom}_{\mathcal{C}_{n-1}}(U_{n-1}, \mathrm{Res}^{\mathcal{Y}_n}_{\mathcal{Y}_{n-1}} W)$$

$$\simeq \mathrm{Hom}_{\mathcal{C}_{n-1}}(U_{n-1}, \mathrm{Res}^{\mathcal{T}_{n-1} \otimes \mathcal{C}_n}_{\mathcal{T}_{n-1} \otimes \mathcal{C}_{n-1}} \mathrm{Res}^{\mathcal{T}_n \otimes \mathcal{C}_n}_{\mathcal{T}_{n-1} \otimes \mathcal{C}_n} W)$$

$$\simeq \mathrm{Hom}_{\mathcal{C}_n}(\mathrm{Ind}^{\mathcal{C}_n}_{\mathcal{C}_{n-1}} U_{n-1}, \mathrm{Res}^{\mathcal{T}_n \otimes \mathcal{C}_n}_{\mathcal{T}_{n-1} \otimes \mathcal{C}_n} W) \simeq \mathrm{Hom}$$

$n = 2k-1$ $\quad \mathcal{C}_n \simeq Q_{2^{k-1}, 2^{k-1}}$

$n-1 = 2k-2$ $\quad \mathcal{C}_{n-1} \simeq \mathrm{Mat}_{2^{k-2}, 2^{k-2}}$

$U_{n-1} \otimes_{\mathcal{C}_{n-1}} \mathcal{C}_n$

$$\downarrow$$

$$\simeq \mathrm{Hom}_{\mathcal{C}_n}(U_n, \mathrm{Res}^{\mathcal{T}_n \otimes \mathcal{C}_n}_{\mathcal{T}_{n-1} \otimes \mathcal{C}_n} W) \simeq \mathrm{Res}^{\mathcal{T}_n}_{\mathcal{T}_{n-1}} \mathrm{Hom}_{\mathcal{C}_n}(U_n, W)$$

n: even

$$\mathbb{R}_{n-1} \circ \left(\mathrm{Res}^{\mathcal{Y}_n}_{\mathcal{Y}_{n-1}} \circ \underbrace{\mathbb{I}_n} \right) \circ \mathbb{R}_n \simeq \underbrace{\mathbb{R}_{n-1} \circ \mathbb{I}_{n-1}} \circ \mathrm{Res}^{\mathcal{T}_n}_{\mathcal{T}_{n-1}} \circ \mathbb{R}_n$$

$$\mathbb{R}_{n-1} \circ \mathrm{Res}^{\mathcal{Y}_n}_{\mathcal{Y}_{n-1}} \circ \cancel{\bullet} \simeq \mathrm{Res}^{\mathcal{T}_n}_{\mathcal{T}_{n-1}} \circ \mathbb{R}_n \oplus \Pi\left(\mathrm{Res}^{\mathcal{T}_n}_{\mathcal{T}_{n-1}} \circ \mathbb{R}_n \right)$$

n: odd

$$\mathbb{I}_{n-1} \mathrm{Res}^{\mathcal{T}_n}_{\mathcal{T}_{n-1}} \oplus \Pi\left(\mathbb{I}_{n-1} \mathrm{Res}^{\mathcal{T}_n}_{\mathcal{T}_{n-1}} \right) \simeq \mathrm{Res}^{\mathcal{Y}_n}_{\mathcal{Y}_{n-1}} \circ \mathbb{I}_n$$

n: even:

$$\boxed{\mathbb{I}_{n+1} \mathrm{Ind}^{\mathcal{T}_{n+1}}_{\mathcal{T}_n} \simeq \mathrm{Ind}^{\mathcal{Y}_{n+1}}_{\mathcal{Y}_n} \circ \mathbb{I}_n}$$

$$\mathrm{Ind}^{\mathcal{T}_{n+1}}_{\mathcal{T}_n} \mathbb{R}_n \oplus \Pi\left(\mathrm{Ind}^{\mathcal{T}_{n+1}}_{\mathcal{T}_n} \mathbb{R}_n \right) \simeq \mathbb{R}_{n+1} \mathrm{Ind}^{\mathcal{Y}_{n+1}}_{\mathcal{Y}_n}$$

n: odd

$$\mathbb{R}_{n+1} \circ \mathrm{Ind}^{\mathcal{Y}_{n+1}}_{\mathcal{Y}_n} \simeq \mathrm{Ind}^{\cancel{\mathcal{Y}}\,\mathcal{T}_{n+1}}_{\mathcal{T}_n} \mathbb{I}_n$$

$$\mathrm{Ind}^{\mathcal{Y}_{n+1}}_{\mathcal{Y}_n} \mathbb{I}_n \simeq \mathbb{I}_{n+1} \mathrm{Ind}^{\mathcal{T}_{n+1}}_{\mathcal{T}_n} \oplus \Pi\left(\mathbb{I}_{n+1} \mathrm{Ind}^{\mathcal{T}_{n+1}}_{\mathcal{T}_n} \right)$$