MTH 309: Introduction to Linear Algebra
Homework 1

Due Tuesday, Feb 12th

Name: ____________________ Person #: __________

1. Determine whether each of the following is a system of linear equations (please write Yes or No next to the equations.)

$$\begin{cases} x + y = 2 \\ x^2 - y = 4 \end{cases} \qquad \begin{cases} xy - 2 = 4 \\ \quad y \ = 3 \end{cases}$$

$$\begin{cases} x = 2 \\ y = 3 \\ z = 5 \end{cases} \qquad \begin{cases} x_1 + 3x_2 - 4x_4 = 7 \\ \qquad\qquad y_1 = x_2 \end{cases} \qquad \begin{cases} a = 2 \\ a = 9 \end{cases}$$

$$x = 4 \qquad \frac{x}{y} = 4$$

2. Write down the augmented matrix of each of the following SLEs (please write it next to the equations.) In addition, describe your answers in the following form: this matrix is a __ × __ matrix.

$$\begin{cases} x_1 - x_2 = 4 \\ 2x_1 + x_3 = 0 \end{cases}$$

$$\begin{cases} -x_1 + x_2 + 4 = 0 \\ \quad x_1 + 3x_2 \ = 3 \end{cases}$$

$$\begin{cases} x_1 + x_2 - 3x_3 = 5 \\ \qquad x_1 + x_2 = 3 - x_3 \end{cases}$$

$$\begin{cases} \qquad x_3 + x_5 = -3 \\ x_1 + x_2 - x_4 = 1 \end{cases}$$

3. Identify what elementary row operation has been performed for each of the following matrices. Please state your answer in one of the following forms:

1) Interchange Row __ and Row __
2) Scaling Row __ by the constant __
3) Scaling Row __ by the constant __, add it to Row __ and replace it

$$\begin{cases} x_1 + 2x_2 + x_3 = 7 \\ \quad\; - x_2 + x_3 = 10 \end{cases} \Longrightarrow \begin{cases} \quad\; - x_2 + x_3 = 10 \\ x_1 + 2x_2 + x_3 = 7 \end{cases}$$

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ -3x_1 + x_2 + 4x_3 = 1 \end{cases} \Longrightarrow \begin{cases} x_1 + x_2 - x_3 = 2 \\ 3x_1 - x_2 - 4x_3 = -1 \end{cases}$$

$$\begin{cases} x_1 + 2x_2 - x_3 = 7 \\ 2x_1 + x_2 - 3x_3 = 10 \end{cases} \Longrightarrow \begin{cases} x_1 + 2x_2 - x_3 = 7 \\ 3x_1 + 3x_2 - 4x_3 = 17 \end{cases}$$

$$\begin{cases} x_1 - x_2 + x_3 = 8 \\ \frac{3}{2}x_1 + x_2 - \frac{3}{4}x_3 = 1 \end{cases} \Longrightarrow \begin{cases} x_1 - x_2 + x_3 = 8 \\ 2x_1 + \frac{1}{2}x_2 - \frac{1}{4}x_3 = 5 \end{cases}$$

(typo corrected)

4. State whether each of the following matrices is in the row echelon form (Please write Yes or No next to the matrices.)

$$\begin{bmatrix} 3 & 2 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 2 & 0 & 1 \\ 0 & \pi & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 0 & 0 \\ 8 & 0 & -2 \end{bmatrix} \qquad \begin{bmatrix} 3 & e & -2 & 3.2 \\ 0 & 0 & \frac{1}{4} & -2 \end{bmatrix}$$

5. Perform *one* elementary row operation to each the following matrices, and convert it into a row echelon form. Please state what operation you performed as you did in Problem 3, and write down the resulting matrix.

1) $\begin{bmatrix} 0 & -2 & 1 & 4 \\ 8 & -1 & 0 & 2 \end{bmatrix}$

2) $\begin{bmatrix} 2 & 3 & 1 \\ -2 & -1 & 0 \end{bmatrix}$

3) $\begin{bmatrix} -4 & 3 & 2 & 7 \\ 0 & 2 & -1 & 8 \\ 0 & 7 & 1 & 6 \end{bmatrix}$

MTH 309: Introduction to Linear Algebra
Homework 2

Due Tuesday, Feb 17th

Name: ____________________ Person #: __________

1. Convert the following matrices into their row echelon forms. Please include all the intermediate matrices at each step of elementary row operation.

a) $\begin{bmatrix} 3 & 2 & -1 \\ 0 & 0 & 4 \\ -2 & 1 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 0 & -1 & 2 \\ 1 & 3 & 2 \\ 2 & 0 & -3 \end{bmatrix}$

c) $\begin{bmatrix} -1 & 2 & -4 \\ 2 & 3 & 1 \\ 1 & -3 & 2 \end{bmatrix}$

2. Given the following matrices in row echelon forms, solve their associated SLE (system of linear equations) by back substitution.

a) $\begin{bmatrix} 4 & -2 & 1 \\ 0 & 1 & -5 \end{bmatrix}$

b) $\begin{bmatrix} 3 & -2 & 1 & 0 \\ 0 & -1 & 3 & 7 \\ 0 & 0 & 2 & 4 \end{bmatrix}$

3. For each of the following augmented matrices, please determine the number of solutions for their associated SLE (system of linear equations.)

a) $\begin{bmatrix} 3 & 9 & -1 & 2 & -2 \\ 0 & 2 & 1 & 3 & 3 \end{bmatrix}$

b) $\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$

c) $\begin{bmatrix} 3 & 4 & 9 \\ 0 & 0 & 0 \end{bmatrix}$

d) $\begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & -4 \end{bmatrix}$

4. Let $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 0 \\ -7 \end{bmatrix}$. (In type format, we use bold fonted letters for vectors, but please use $\overrightarrow{v}$ and $\overrightarrow{w}$ in your writing.) Compute the following:

a) $3\mathbf{v}$

b) $\mathbf{v} + \mathbf{w}$

c) $\frac{1}{2}\mathbf{v} + 2\mathbf{w}$

5. Describe the solution set of the following SLEs, as the spanning set of certain vectors. In other words, give a collection of vectors that span the solution set.

a) $\begin{bmatrix} 3 & 1 & 0 \end{bmatrix}$

b) $\begin{bmatrix} -2 & 1 & 3 & 0 \\ 0 & 2 & -4 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 3 & -1 & -2 & 0 & 0 \\ 0 & 1 & 4 & -1 & 0 \end{bmatrix}$

6. Given $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$, set up (don't solve!) an SLE with variables x_1, x_2 such that

$$\mathbf{u} = x_1\mathbf{v} + x_2\mathbf{w}$$

7. Given $\mathbf{v} = \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 6 \\ 0 \\ -\frac{1}{2} \end{bmatrix}$, set up (don't solve!) an SLE with variables x_1, x_2 such that

$$\mathbf{u} = x_1\mathbf{v} + x_2\mathbf{w}$$

8. Given $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} -3 \\ 6 \\ -12 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, write $\mathbf{u}$ as a linear combination of $\mathbf{v}$ and $\mathbf{w}$ in <u>two different ways</u>.

9. Determine if the following vectors are linearly independent. In other words, determine if the zero vector can be written as a linear combination of them in only one unique way.

a) $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

b) $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 5 \\ 5 \\ -2 \end{bmatrix}$

c) (For people who want a little challenge.) In problem b), pick one of the vectors, and write it as a linear combination of the others.

MTH 309: Introduction to Linear Algebra
Homework 3

Due Tuesday, Feb 26th

Name: ____________________ Person #: __________

Instructions: please print the pages *single-sided* and use no staples. Please write your name on top of each page.

1. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

1) Determine if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

2) Please write $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in <u>two different ways</u>.

(2. continued)

2. Given a linear map $f : \mathbb{R} \to \mathbb{R}^2$, if $f(1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, please determine $f(3.5)$.

3. Given a linear map $f : \mathbb{R}^2 \to \mathbb{R}$, if

$$f(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = -\frac{1}{2}, \qquad f(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = 4,$$

please determine $f(\begin{bmatrix} 1 \\ 1 \end{bmatrix})$ and $f(\begin{bmatrix} 2 \\ 3 \end{bmatrix})$.

4. Given a linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$, if

$$f(\begin{bmatrix}1\\2\end{bmatrix}) = -3, \qquad f(\begin{bmatrix}-2\\1\end{bmatrix}) = 1,$$

Please determine $f(\begin{bmatrix}3\\4.5\end{bmatrix})$.

5. Compute the following multiplication of matrices.

1) $\begin{bmatrix}3 & -1\end{bmatrix} \cdot \begin{bmatrix}2\\-2\end{bmatrix}$

2) $\begin{bmatrix}0 & -1.5 & 2 & 3\end{bmatrix} \cdot \begin{bmatrix}\pi\\2\\-1\\3\end{bmatrix}$

3) $\begin{bmatrix}a_1 & a_2 & a_3\end{bmatrix} \cdot \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ (of course, you answer may have variables in it.)

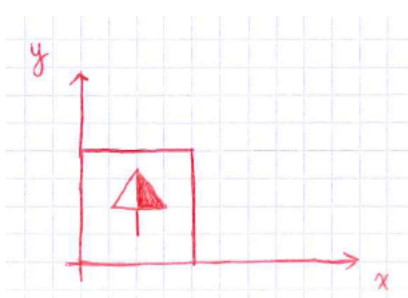
6. Compute the following multiplication of matrices.

1) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

2) $\begin{bmatrix} 1 & -1 & 8 \\ 2 & 3 & 7 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 10 \\ 1 \end{bmatrix}$

3) $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ (of course, you answer may have variables in it.)

7. Suppose the following picture is located at a unit square in the xoy-plane. Draw its image in the xoy-plane under the following linear transformation. Here, $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



1) $f(\mathbf{v}) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{v}$. Please compute $f(\mathbf{e}_1)$ and $f(\mathbf{e}_2)$, then draw the image.

2) $g(\mathbf{v}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{v}$. Please compute $f(\mathbf{e}_1)$ and $f(\mathbf{e}_2)$, then draw the image.

3) $h(\mathbf{v}) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \mathbf{v}$. Please compute $f(\mathbf{e}_1)$ and $f(\mathbf{e}_2)$, then draw the image.

8. Given $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{w}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

1) Write $\mathbf{e}_1$ as a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$. Also write $\mathbf{e}_2$ as a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$.

2) If a linear map f has the property $f(\mathbf{w}_1) = \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$ and $f(\mathbf{w}_2) = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$. Compute $f(\mathbf{e}_1)$, $f(\mathbf{e}_2)$, and write down the matrix associated with f.

MTH 309: Introduction to Linear Algebra
Homework 4

Due Tuesday, Mar 5th

Name: ____________________ Person #: __________

Instructions: please print the pages *single-sided* and use no staples. Please write your name on top of each page.

Python problems

Instructions: please code in Jupyter and save your codes as a pdf file (use the "save as .pdf via LaTex" option), then upload it on UBLearns.

1. Enter the following matrices

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 3 & 3 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 9 & 10 & -1 & -3 & -1 \end{bmatrix}, \qquad C = \begin{bmatrix} -1 \\ 0 \\ 4 \\ 7 \end{bmatrix}$$

2. Find the reduced row echelon form of the following matrices

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 7 & -4 & 1 \\ 3 & -2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & -2 & 9 & 3 \\ 10 & -2 & 5 & 0 \\ 4 & 3 & 0 & 12 \end{bmatrix}$$

3. Convert the following matrices into its row echelon forms *manually*, i.e perform necessary elementary row operations step by step and show all intermediate matrices along the way.

$$A = \begin{bmatrix} 4 & 9 \\ 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 4 & 9 \\ 2 & 4 & -1 \\ 4 & -3 & 0 \end{bmatrix}$$

4. Calculate the following in Python.

$$\begin{bmatrix} 3 & 3 & -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ \frac{2}{3} \\ -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ \frac{2}{3} \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 3 & -1 & 4 \end{bmatrix}$$

Written/Python hybrid problems

Instructions: you may use Python to work out (part of) the problem, write down only the final answer, and submit it in print. No codes or steps are needed on the physical copy.

5. 1) Given

$$\mathbf{v} = \begin{bmatrix} \frac{3}{11} \\ \frac{82}{3} \\ 2 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{20} \\ 1 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} \frac{89}{154} \\ \frac{3343}{420} \\ \frac{11}{7} \end{bmatrix}$$

Write $\mathbf{u}$ as a linear combination of $\mathbf{v}$ and $\mathbf{w}$.

2) If $f : \mathbb{R}^2 \to \mathbb{R}$ is a linear map, $f(\mathbf{v}) = 3$, $f(\mathbf{w}) = 4$, compute $f(\mathbf{u})$.

6. If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map, $\mathbf{e}_1$ and $\mathbf{e}_2$ are the standard basis vector for $\mathbb{R}^2$,

$$f(\mathbf{e}_1) = \begin{bmatrix} 20 \\ 31 \end{bmatrix}, \qquad f(\mathbf{e}_2) = \begin{bmatrix} -41 \\ -12 \end{bmatrix}$$

compute

$$f(\begin{bmatrix} 2 \\ 3 \end{bmatrix}) = \qquad f(\begin{bmatrix} \frac{2}{7} \\ \frac{1}{3} \end{bmatrix}) = \qquad f(\begin{bmatrix} -249 \\ 348 \end{bmatrix}) =$$

7. Given $\mathbf{v} = \begin{bmatrix} 5 \\ -11 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$. If

$$f(\mathbf{v}) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \qquad f(\mathbf{w}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

Write down the matrix associated to f using the standard basis $\mathbf{e}_1$, $\mathbf{e}_2$.

Written problems

Instructions: please include all steps in writing.

8. Convert the following matrices to their reduced row echelon forms.

1) $\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -3 \end{bmatrix}$

2) $\begin{bmatrix} 3 & 4 & 7 \\ 0 & -1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

9. Find the inverses of the following matrices.

1) $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

2) $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$

3) $\begin{bmatrix} 1 & 4 & 3 \\ 2 & 9 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

MTH 309: Introduction to Linear Algebra
Homework 5

Due Tuesday, Mar 26th

Name: ____________________ Person #: __________

Instructions: please print the pages *single-sided* and use no staples. Please write your name on top of each page.

1. Invert the following elementary matrices directly.

a) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

2. For each of the following matrices, write it as a product as elementary matrices. Also write its inverse as a product as elementary matrices.

a) $\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 7 \end{bmatrix}$

4. Determine if the following matrix is invertible.

$$\begin{bmatrix} -2 & -1 & 2 \\ 0 & 4 & 9 \\ -1 & 2 & -1 \end{bmatrix}$$

5. Given the three vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -3 \\ 9 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 7 \\ 2 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -8 \\ 17 \\ 64 \end{bmatrix}$$

determine if they are linearly independent using determinants.

6. Use Cramer's law, compute the inverse of the following matrices using determinants.

a) $\begin{bmatrix} 8 & -2 \\ 1 & 4 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 5 & 1 \\ -3 & -1 & 0 \\ 5 & 2 & 7 \end{bmatrix}$

7. Given the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$. Suppose

$$\begin{aligned} \mathbf{c} &= \mathbf{a} - 3\mathbf{b} \\ \mathbf{d} &= \mathbf{c} + \mathbf{b} + 2\mathbf{a} \\ \mathbf{e} &= \mathbf{c} + 2\mathbf{d} \end{aligned}$$

1) Write $\mathbf{e}$ as a linear combination of $\mathbf{a}$, $\mathbf{b}$.

2) Write $\mathbf{d}$ as a linear combination of $\mathbf{a}$, $\mathbf{c}$.

3) Determine if $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$ are linearly independent.

4) Suppose $\mathbf{a}$ and $\mathbf{b}$ are linearly independent. Determine the dimension of $\operatorname{span}\langle \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e} \rangle$.

8. Given

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -9 \\ -6 \\ 3 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 8 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ \frac{7}{2} \\ 10 \end{bmatrix}$$

Find a maximal collection of linearly independent vectors and state the dimension of the vector space $\operatorname{span}\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \rangle$.

9. Given

$$\begin{bmatrix} 4 & 6 & -2 & 1 \\ 8 & 12 & -4 & 2 \\ 1 & 0 & -3 & 9 \\ 24 & 42 & -2 & -29 \end{bmatrix}$$

Determine the rank of A and the dimension of the column space of A. Also give a minimal collection of vectors which span $\operatorname{col} A$

MTH 309: Introduction to Linear Algebra
Homework 6

Due Tuesday, Apr 9nd

Name: ____________________ Person #: __________

Instructions: please print the pages *single-sided* and use no staples. Please write your name on top of each page.

1. Given

$$A = \begin{bmatrix} 1 & 5 & 10 \\ -2 & -10 & -20 \end{bmatrix}$$

1) Determine Rank A
2) Write Row A (the row space of A) as the span of one or several vectors.

3) Write Null A (the null space of A) as the span of one or several vectors.

4) Check if the three combined vectors from 2) and 3) are linearly independent.

2. Given

$$A = \begin{bmatrix} 2 & 3 & 1 & -2 \\ -1 & 0 & 7 & -1 \\ -9 & -3 & 48 & -5 \end{bmatrix}$$

1) Determine Rank A

2) Write the Row A (the row space of A) as the span of one or several vectors.

3) Write the Null A (the null space of A) as the span of one or several vectors.

4) Check if the four combined vectors from 2) and 3) are linearly independent.

3. Given

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$$

Use matrix inversion, find $[\mathbf{w}]_{\{\mathbf{v}_1,\mathbf{v}_2\}}$. In other words, write down the coordinates of $\mathbf{w}$ under the basis $\mathbf{v}_1, \mathbf{v}_2$ in the form of a vector. Please show all work of inverting a matrix.

4. Given

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 8 \\ 3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 4 \\ -7 \\ 5 \end{bmatrix}$$

Use matrix inversion, find $[\mathbf{w}]_{\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}}$. In other words, write down the coordinates of $\mathbf{w}$ under the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in the form of a vector. Please show all work of inverting a matrix.

5. Given

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{v}_1 = \begin{bmatrix} 8 \\ -3 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Find:
1) The change-of-coordinates matrix from $\{\mathbf{v}_1, \mathbf{v}_2\}$ to $\{\mathbf{e}_1, \mathbf{e}_2\}$.
2) The change-of-coordinates matrix from $\{\mathbf{e}_1, \mathbf{e}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$.

6. Given

$$\mathbf{w}_1 = \begin{bmatrix} 4 \\ 7 \end{bmatrix} \quad \mathbf{w}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \mathbf{v}_1 = \begin{bmatrix} -2 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Find the change-of-coordinates matrix from $\{\mathbf{v}_1, \mathbf{v}_2\}$ to $\{\mathbf{w}_1, \mathbf{w}_2\}$.

7. Determine if the following vectors form a basis of $\mathbb{R}^2$. Please provide a brief justification. You may omit the details of your calculations.

1) $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$

2) $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2.5 \\ -10 \end{bmatrix} \right\}$

3) $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}$

4) $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

5) $\left\{ \begin{bmatrix} 10 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

8. Determine if the following vectors form a basis of $\mathbb{R}^3$. Please provide a brief justification. You may omit the details of your calculations.

1) $\left\{ \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -12 \\ 4 \end{bmatrix} \right\}$

2) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} \right\}$

MTH 309: Introduction to Linear Algebra
Homework 7

Due Tuesday, April 9th

Name: ____________________ Person #: __________

Instructions: please print the pages *single-sided* and use no staples. Please write your name on top of each page. Please submit all pages even if some of them were left undone.

1. Given $\{\mathbf{w}_1, \mathbf{w}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_2\}$ two bases of $\mathbb{R}^2$, if

$$\begin{aligned}\mathbf{w}_1 &= 6\mathbf{v}_1 + \mathbf{v}_2 \\ \mathbf{w}_2 &= \mathbf{v}_1 + 2\mathbf{v}_2\end{aligned}$$

a) Find the change-of-coordinate matrix from $\{\mathbf{w}_1, \mathbf{w}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$.
b) Find the change-of-coordinate matrix from $\{\mathbf{v}_1, \mathbf{v}_2\}$ to $\{\mathbf{w}_1, \mathbf{w}_2\}$.
c) If $\mathbf{u}_{\{\mathbf{v}_1,\mathbf{v}_2\}} = \begin{bmatrix}3\\1\end{bmatrix}$, find $\mathbf{u}_{\{\mathbf{w}_1,\mathbf{w}_2\}}$.
d) If $\mathbf{t}_{\{\mathbf{w}_1,\mathbf{w}_2\}} = \begin{bmatrix}-2\\4\end{bmatrix}$, find $\mathbf{t}_{\{\mathbf{v}_1,\mathbf{v}_2\}}$.

2. Given

$$\mathbf{w}_1 = \begin{bmatrix}3\\5\end{bmatrix} \quad \mathbf{w}_2 = \begin{bmatrix}-2\\1\end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix}-1\\4\end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix}0\\2\end{bmatrix}$$

Find the change-of-coordinate matrix from $\{\mathbf{w}_1, \mathbf{w}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$.

3. If $P = \begin{bmatrix}4 & -1\\-7 & 2\end{bmatrix}$ is the change-of-coordinate matrix from $\{\mathbf{w}_1, \mathbf{w}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map whose defining matrix under $\{\mathbf{w}_1, \mathbf{w}_2\}$ is $\begin{bmatrix}3 & 1\\2 & 10\end{bmatrix}$, that is,

$$f(\mathbf{u})_{\{\mathbf{w}_1,\mathbf{w}_2\}} = \begin{bmatrix}3 & 1\\2 & 10\end{bmatrix} \mathbf{u}_{\{\mathbf{w}_1,\mathbf{w}_2\}}$$

a) If $\mathbf{u}_{\{\mathbf{w}_1,\mathbf{w}_2\}} = \begin{bmatrix}4\\1\end{bmatrix}$, compute $f(\mathbf{u})_{\{\mathbf{w}_1,\mathbf{w}_2\}}$.

b) If $\mathbf{u}_{\{\mathbf{w}_1,\mathbf{w}_2\}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, compute $f(\mathbf{u})_{\{\mathbf{e}_1,\mathbf{e}_2\}}$. Here, $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

c) If $\mathbf{s}_{\{\mathbf{e}_1,\mathbf{e}_2\}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, compute $f(\mathbf{s})_{\{\mathbf{w}_1,\mathbf{w}_2\}}$.

d) If $\mathbf{s}_{\{\mathbf{e}_1,\mathbf{e}_2\}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, compute $f(\mathbf{s})_{\{\mathbf{e}_1,\mathbf{e}_2\}}$.

MTH 309: Introduction to Linear Algebra
Homework 8

Due Tuesday, April 23th

Name: ____________________ Person #: __________

Recitation time (choose one)

Tue 8 am __________

Tue 4 pm __________

Thur 8 am __________

Instructions: please print the pages *single-sided* and use no staples. Please write your name on top of each page. Please submit all pages even if some of them were left undone. All problems should be worked out by hand unless Python is suggested.

1. Given the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ Find all its eigenvalue(s). You may use the quadratic formula.

2. Determine the number of eigenvalues for each of the following matrices. Please include your calculations.

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 7 \\ 0 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

3. Given the matrix $A = \begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$. Find all its eigenvalue(s) and describe the corresponding eigenspace(s). Does A have 2 linearly independent eigenvectors? Why?

4. Given the matrix $A = \begin{bmatrix} -8 & 10 \\ -5 & 7 \end{bmatrix}$. Find all its eigenvalue(s) and describe the corresponding eigenspace(s). Does A have 2 linearly independent eigenvectors? Why?

5. Describe the 3-eigenspace (the eigenspace with eigenvalue 3) of the following matrix as the span of one or few vectors.

$$A = \begin{bmatrix} \frac{10}{3} & -\frac{2}{3} & -1 \\ -\frac{1}{3} & \frac{11}{3} & 1 \\ \frac{2}{3} & -\frac{4}{3} & 1 \end{bmatrix}$$

6. Find the characteristic polynomial of the following matrix.

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & -2 \\ -1 & 0 & 4 \end{bmatrix}$$

7. Suppose Alice decides to study each of the subjects every night: maths, physics, chemistry. Suppose this stochastic event has the following matrix

	Maths	Physics	Chemistry
Maths	.3	.3	.4
Physics	.5	.1	.4
Chemistry	.2	.6	.2

Suppose the state vector today is $\mathbf{v} = \begin{bmatrix} .3 \\ .2 \\ .5 \end{bmatrix}$.

1) Find the state vector tomorrow.

2) Find the state vector in 10 days. (You may use Python for part of the computation. No coding process needs to be shown here.)

3) Use Python (and possibly finish by a hand calculation), find the stable-state vector. Your answer should have rational entries rather than floating point entries.

MTH 309: Introduction to Linear Algebra
Homework 9

Due Tuesday, April 30th

Name: ____________________ Person #: __________

Recitation time (choose one)

Tue 8 am __________

Tue 4 pm __________

Thur 8 am __________

Instructions: please print the pages *single-sided* and use no staples. Please write your name on top of each page. Please submit all pages even if some of them were left undone. All problems should be worked out by hand unless Python is suggested.

1. Use Python and diagonalize the following matrices **over the real numbers**. Please write down the command you use, state if each of them is diagonalizable and/or the diagonalization (the steps of inputting data can be omitted.)

$$A = \begin{bmatrix} 3 & 7 \\ -1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 100 & -27 \\ 204 & -49 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 9 & -1 \\ 2 & -4 & 1 \\ 6 & 2 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 2 & -2 \\ 4 & 5 & -4 \\ 4 & 4 & -3 \end{bmatrix}$$

2. Use Python and find eigenvalues and eigenvectors for the following matrices. Please state the multiplicity of each eigenvalue. Without using the diagonalization command, is each of the matrix diagonalizable and why? Please write down the command you use and the final answer (the steps of inputting data can be omitted.)

$$A = \begin{bmatrix} 32 & -10 \\ 78 & -24 \end{bmatrix} \qquad B = \begin{bmatrix} 97 & -26 \\ 195 & -46 \end{bmatrix} \qquad C = \begin{bmatrix} 13 & 6 & -10 \\ 4 & 7 & -4 \\ 12 & 8 & -9 \end{bmatrix} \qquad D = \begin{bmatrix} -1 & -4 & 4 \\ -8 & -5 & 8 \\ -8 & -8 & 11 \end{bmatrix}$$

3. Given eigenvalues 2 and 3, diagonalize the following matrix.

$$A = \begin{bmatrix} 8 & -2 \\ 15 & -3 \end{bmatrix}$$

4. Given eigenvalues 3 and 1, determine if the following matrix is diagonalizable.

$$B = \begin{bmatrix} 5 & 3 & -4 \\ 4 & 5 & -4 \\ 6 & 5 & -5 \end{bmatrix}$$

5. Given a linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$, if $\mathbf{V} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of $\mathbb{R}^3$, and further,

$$f(\mathbf{v}_1) = 3\mathbf{v}_1 \qquad f(\mathbf{v}_2) = -\mathbf{v}_2 \qquad f(\mathbf{v}_3) = 9\mathbf{v}_3$$

a) State the defining matrix of f under the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

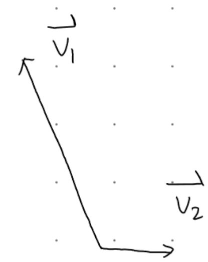
b) Let $\mathbf{W} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ be another basis of $\mathbb{R}^3$ and $P = \begin{bmatrix} -3 & 2 & 0 \\ -4 & -1 & -2 \\ 0 & 1 & -1 \end{bmatrix}$ be the change-of-coordinate matrix from $\mathbf{V}$ to $\mathbf{W}$. Let A be the defining matrix for f under the basis $\mathbf{W}$, diagonalize A.

6. Given a linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$, if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of $\mathbf{R}^2$, and

$$f(\mathbf{v}_1) = 2\mathbf{v}_1 \qquad f(\mathbf{v}_2) = -\mathbf{v}_2$$

Given the vector $\mathbf{w} = \mathbf{v}_1 + 3\mathbf{v}_2$, draw $\mathbf{w}$, $f(\mathbf{w})$ and $f^2(\mathbf{w}) = f(f(\mathbf{w}))$ on the last page.

HW #9



D10.

MTH 309: Introduction to Linear Algebra
Homework 10

Due Tuesday, May 7th

Name: ____________________ Person #: __________

Recitation Time (circle one)	Tue 8 am	Tue 4 pm	Thur 8 am

Volunteer[*]		Print (circle one)	single-sided	double-sided

Instructions: please use no staples. It is preferable if you print the pages single-sided. If some problems are left as blank, submit the pages nevertheless. All problems should be worked out by hand unless Python is suggested.

1. Given the following

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \qquad \mathbf{v}_0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

1) Use the iteration algorithm, start with $\mathbf{v}_0$ and perform three iterations. Please state the vector at the end of each iteration and its difference with $\mathbf{w}$.

[*]Check if you are willing to have your solutions posted anonymously. Once selected, I will erase your name before posting, and you will get a small bonus on your HW score.

2) Choose your own initial vector, use Python and compute the vector at the end of the 10-th iteration. Compute its difference with $\mathbf{w}$. Please state the initial vector, the command you used, and the final answer.

2. Given

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

compute $\cos\theta$, where θ is the angle between the two vectors.

3. Given

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

1) Find the projection of $\mathbf{w}$ along the direction of $\mathbf{v}$.

2) Find the projection of $\mathbf{v}$ along the direction of $\mathbf{w}$.

4. Determine if each of the following set is orthogonal.

1)

$$\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

2)

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

3)

$$\left\{ \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} \right\}$$

4)

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix} \right\}$$

5. Use the Gram-Schmidt process, construct an orthogonal basis from the following set

1)

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

2)

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix}$$

6. For the newly defined $\star$ operation below, find a counterexample (i.e. vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}$) and show that the following property fails for your choice of vectors.

$$(\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{w}_1 = \mathbf{v}_1 \cdot \mathbf{w}_1 + \mathbf{v}_2 \cdot \mathbf{w}_1$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \star \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 + y_2$$