① Def. A ~~Linear~~ System of Linear Equations (SLE) is a collection of equations whose terms are in the form of a constant $\cdot$ variable
times
and constant terms

Remark. (x^2, $\sqrt{x}$, $\sin x$, $\frac{1}{x}$ may not appear.
SLE may have redundant info, may have contradicting info)

② Def. A matrix with real numbers as entries is a list of real numbers arranged into a rectangle. An $m \times n$ matrix has m rows and n columns.
by

③ Def. The augmented matrix of an SLE is the matrix whose entries are the coefficients and constant terms of the SLE, arranged in a specific order.

④ Def. A matrix is in ~~the~~ a Row Echelon Form if the following conditions are satisfield:

1) ~~All zero~~ A row with all zero entries sit at the bottom

2) The leading nonzero entry sits to the left of the the leading nonzero entry in the next row.

⑤ Elementary Row Operations on a matrix are the following:

1) Scaling: multiply every entry in a row by a nonzero constant.

2) Interchange: Swap the position of two rows

3) Replacement: multiply a constant to all entries in a row, add it to a different row and replace it.

(6) Remark, one can use an elementary row operation to change a leading nonzero entry and make it zero (to "kill" it)

the constant is given by

$$\frac{\text{negative of the leading nonzero entry you want to "kill"}}{\text{leading nonzero entry of the row you use \& fix}}$$

(7) Def. Pivots are the leading nonzero entries in each row in a row echelon form

~~7 A vector is a ma~~

(8) Row reduction algorithm:

Step 1: Identify one ~~lef~~ row whose pivot is the leftmost among all rows

Step 2: Use its pivot to "kill" all entries beneath it

Step 3: Ignore row 2, treat the remaing rows as a new problem & start over

(9) Remark: ~~row op~~ elementary operations do not change the solutions of a SLE

(10) <u>Summary</u>. after ignoring all-0 rows in the row echelon form

When # variables = # equations.

SLE has 1 solutions

When # variables > # equations

SLE has infinitely many solutions

10 cont.) when # variables $<$ # equations

SLE has no solutions

(11) Def vectors are matrices with a single column
the height of a vector is the number of rows

(12) addition of vectors is adding the entries component-wise

by multiplying a ~~constant~~ vector = ~~vector whose entries are constant~~
with a scalar, each of the entries get multiplied by that constant

(13) A linear combination of vectors $\vec{v}$ & $\vec{w}$
is the result of performing addition & scalar multiplication
many times

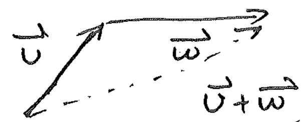
(14) span $\langle \vec{v}, \vec{w} \rangle$ is the collection of vectors that are linear combinations of $\vec{v}$ and $\vec{w}$

* updated 2/5

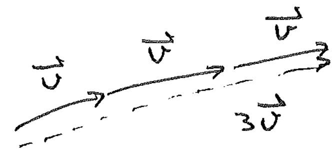
(15) ~~Recall: if we at.~~

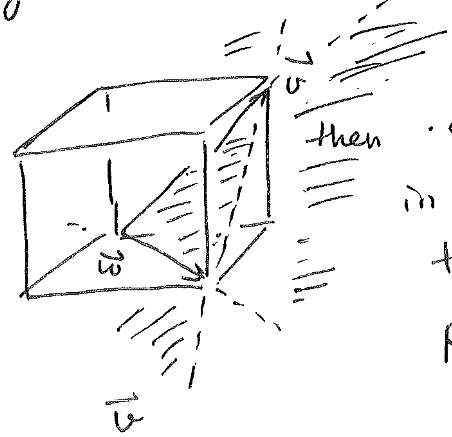
Geometric intuition / Motivation

if $\vec{v} + \vec{w}$ is given by



& $3\vec{v}$:





then · Span $\langle \vec{v}, \vec{w} \rangle$ consists of all vectors in the shaded plane, extend beyond the border to infinity

Remark: $\vec{0}$ is in the span of any # of any vectors

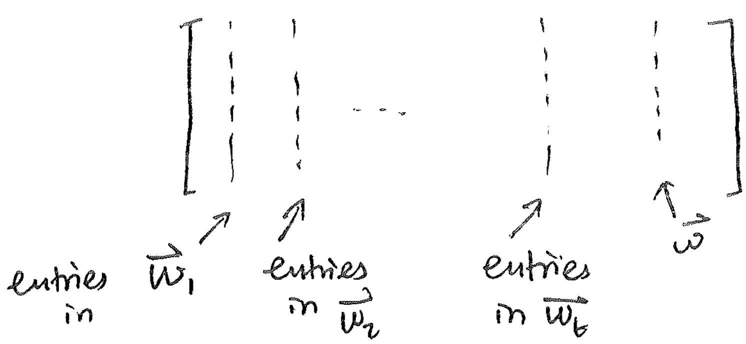
(16) To determine if $\vec{w}$ is a linear combination of

$\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_t$ (t vectors in total)

it is enough to find real numbers $x_1, \cdots, x_t$

such that $\vec{v} = x_1 \vec{w_1} + \cdots + x_t \vec{w_t}$

last time, we showed $x_1, \cdots, x_n$ is the solution for the SLE with augmented matrix



we denote this matrix as $[\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_t, \vec{w}]$

Kaution: this is NOT a matrix with a single row but with as many rows as the height of $\vec{w}$

(17) Goal: describe the solutions when we have infinitly many solutions for a SLE

we write $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_t \end{bmatrix}$ for the solution $x_1, \ldots, x_t$ of an SLE in the <u>vector form</u>

(t: # of variables)

(18) <u>Def</u>: A SLE is said to be <u>homogeneous</u> if the constant terms are zero

(i.e. the right-most column has all zero entries, or equivalently, in the eqs. all terms have deg 1)

~~Remark~~

Remark:

(19) The solution set may be described in different ways using a ~~different~~ combination of different vectors, but the # of spanning vectors are always the same, if the vectors are "well-chosen to be minimal". In this case, the # of spanning vectors is defined to be the dimension of the solution set (space)

also, observe

dimension of the solution space = # var − # eq (in row echelon form, after deleting zeros)

(20) More precise definition of "well-chosen to be minimal".

Naive version: none of the vectors can be written as a linear combination of the others.

Def ~~For~~ Vectors $\vec{v_1}, \dots, \vec{v_n}$ are said to be <u>linearly independent</u>

(19) Formal def (from each other) if the vector equation

(20) cont.

$$x_1\vec{v_1} + x_2\vec{v_2} + \dots + x_n\vec{v_n} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

has a unique solution. We call $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ the zero vector, denoted as $\vec{0}$.

~~Def A SLE is called homogeneous if the constants terms are all zero (i.e. the right-most column in the augmented matrix is a zero vector)~~

(15) Def.

(21) Remark: if # vectors > height
then they are guaranteed to be linearly <u>dependent</u>

(* updated 2/7)

(22) Recall, if we want to write $\vec{w}$ as a linear combination of $\vec{v}_1, \dots, \vec{v}_n$

(optional) that is the same problem as finding real numbers $x_1, \dots, x_n$

such that $$\vec{w} = x_1\vec{v}_1 + \dots + x_n\vec{v}_n \quad (1)$$

i.e. finding solutions to the SLE ~~with~~ represented by the augmented matrix

$$\left[\begin{array}{ccc|c} \vdots & \cdots & \vdots & \vdots \\ \vdots & & \vdots & \vdots \end{array}\right]$$

written as $[\vec{v}_1, \dots, \vec{v}_n, \vec{w}]$

entries of $\vec{v}_1$ — entries of $\vec{v}_n$ — entries of $\vec{w}$

If we can write it differently,

$$\vec{w} = y_1\vec{v}_1 + \dots + y_n\vec{v}_n \quad (2)$$

then subtract (2) from (1)

$$0 = (y_1 - x_1)\vec{v}_1 + \dots + (y_n - x_n)\vec{v}_n$$

then $y_1 - x_1, \dots, y_n - x_n$ are the solutions to

$$[\vec{v}_1, \dots, \vec{v}_n, \vec{0}]$$

if $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent,

such SLE has a unique solution $\vec{0}$

$$y_1 - x_1 = 0, \dots, y_n - x_n = 0$$

$$y_1 = x_1, \dots, y_n = x_n$$

Conclusion. The expression of writing $\vec{w}$ as a linear combination of $\vec{v}_1, \dots, \vec{v}_n$ is unique, if $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent.

(23) Def A vector space is the spanning of a collection of vectors (could be infinitely many, but we are going to focus on just the finite case)

(24) Def If the spanning vectors are linearly independent, they are called a basis of the vector space they span

the number of vectors is called the dimension of the vector space, denoted by dim.

(24.5) Def. in $\mathbb{R}^n$, $\vec{e}_1 = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}$, ... $\vec{e}_n = \begin{bmatrix}0\\\vdots\\1\end{bmatrix}$ is called ~~a~~ the standard basis of $\mathbb{R}^n$

(25) Remark. A basis of a vector space is NOT UNIQUE.

The solution set of a SLE is always a vector space.

(26) Def (At the moment) we will denote an n-dimensional vector spaces by $\mathbb{R}^n$

(27) Def. $f: \mathbb{R}^n \to \mathbb{R}^m$ a map between vector spaces is an assignment which assigns to any n-dimensional vector (an input), an m-dimensional vector (output, or image)

~~Def. A map between vector spaces is Linear, if it is unique determined by its image on the a basis of $\mathbb{R}^n$, in the following way,~~

~~$\mathbb{R}^n = \text{span}\langle \vec{v}_1, \dots, \vec{v}_n \rangle$ (i.e. $\vec{v}_1, \dots, \vec{v}_n$ is a basis of $\mathbb{R}^n$)~~

~~then $f(v$~~

~~$f(a_1\vec{v}_1 + \dots + a_n\vec{v}_n) = a_1 f(\vec{v}_1) + \dots + a_n f(\vec{v}_n)$~~

~~for any constant~~

Def. A map between vector spaces is linear, if it is uniquely
(28) determined by its image on a basis of $\mathbb{R}^n$

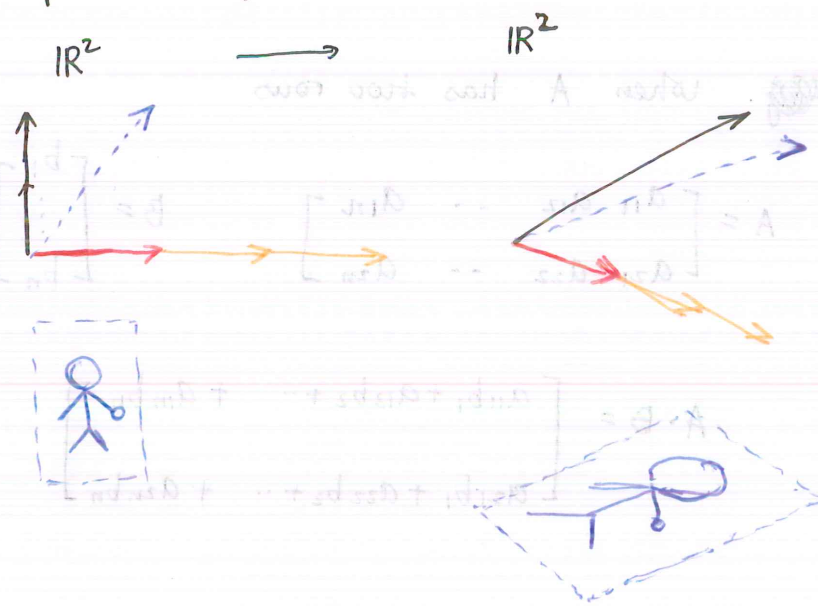
i.e. if $\vec{v}_1, \ldots, \vec{v}_n$ is a basis of $\mathbb{R}^n$

then there is enough information to recover f by knowing $f(\vec{v}_1), \ldots, f(\vec{v}_n)$:

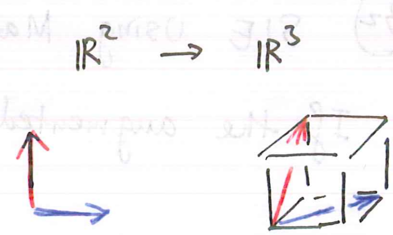
$$f(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) = a_1 f(\vec{v}_1) + a_2 f(\vec{v}_2) + \cdots + a_n f(\vec{v}_n)$$

(28.5) Alternatively, f is a linear map if $f(\vec{v}+\vec{w}) = f(\vec{v}) + f(\vec{w})$, $f(c\vec{v}) = c f(\vec{v})$

(29) Geometric Interpretation of linear transformations (linear maps)



LINEAR TRANSFORMATION USING MATRICES

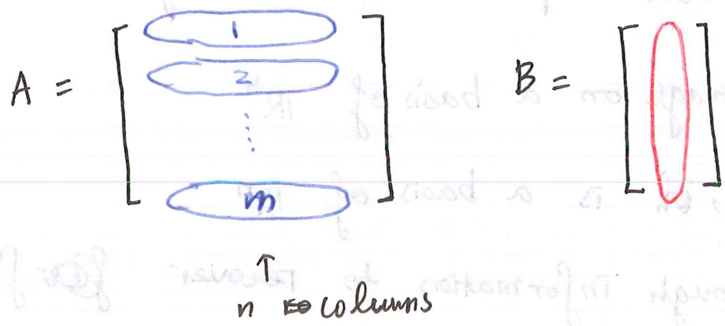


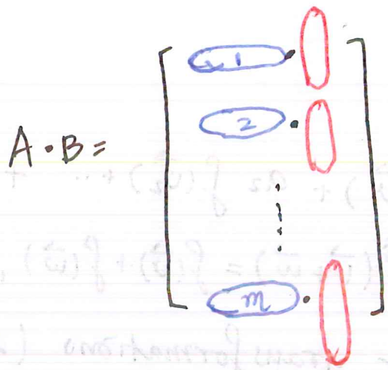
(30) $A = [a_1 \; a_2 \; \cdots \; a_n]$ $\qquad B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

$$A \cdot B = [a_1 b_1 + a_2 b_2 + \cdots + a_n b_n]$$

a one by one matrix

(31)





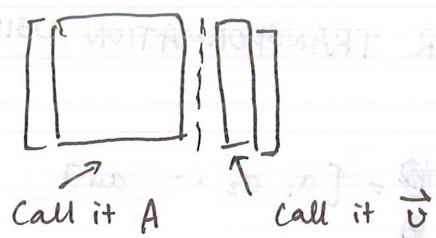
~~eg~~. when A has two rows

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + \cdots + a_{1n}b_n \\ a_{21}b_1 + a_{22}b_2 + \cdots + a_{2n}b_n \end{bmatrix}$$

(32) SLE using Matrix Multiplication.

If the augmented matrix looks like



then the SLE : $A \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{v}$

(33) **Fact** Maps that are given as Left Multiplication by a matrix is always linear.

(34) The Matrix associated to a linear map is the matrix

$$[\, f(\vec{e_1}),\ f(\vec{e_2}) \ \cdots \ f(\vec{e_n}) \,]$$

(35) Remark. Multiplication by ~~xo~~ a matrix is linear AND vice versa, every linear map is in the form as multiplication by a matrix

(~~✗ updated 2/18~~)

(36) Remark

the matrix $\begin{bmatrix} k & \\ & 1 \end{bmatrix}$ has the effect of horizontal stretch by a factor of k

$\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$ has the effect of ~~flip~~, reflection along the x-axis

- $\begin{bmatrix} 1 & k \\ & 1 \end{bmatrix}$ entries off the diagonal have the effect of shearing i.e. rotate / stretch

$\begin{bmatrix} 1 & \\ & \end{bmatrix}$ projection to the x-axis

Lecture Notes 2/21 Thur 11 am – 12:20 pm NSC 228

(37) Def. $I_n = \begin{array}{c} n \text{ cols} \\ \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \end{array} n \text{ rows}$ (0's at empty spots)

is the identity $n \times n$ matrix. (I if the size is clear from context)

<u>Remark</u>. In terms of pictures, the linear map

$f(\vec{v}) = I_n \cdot \vec{v}$

preserves the vectors & preserve any picture in the domain

(38) e.g. $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ $B = \cancel{\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}}$ $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

Check: $AB = \begin{array}{c c} 1 \\ 2 \end{array}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \cdot \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1\;1 & 1\;2 \\ 2\;1 & 2\;2 \end{bmatrix}$

similarly $BA = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$

We call A & B are <u>inverses</u> of each other

<u>Def</u> If $AB = BA = I$, then A & B are called <u>the</u> inverse matrices of each other.

Remark. The inverse is unique!

~~If $AB = I$ & $AB' = I$ $A(B-B') =$~~ (we will see in a future lecture)

(39) ~~The defn~~ e.g

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are in reduced row-echelon form

---

matrices NOT in reduced row-echelon form

pivot NOT 1

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

entry above pivot NOT zero

A matrix in row echelon form is said to be further in reduced in row echelon form if

1. all pivots are 1
2. entries above pivots are also zero

(do example # 29 b from ~~last~~ ~~end~~ the page) in the end)

(40) Summarize:

Row reduction algorithm, Revisted.

① Put the matrix in row echelon form

② ~~use~~ Rescale each row so the pivots become all 1's

③ Use the pivots to "kill" the entries above each pivot.

Find the inverse of a matrix.

(Do example # 30 from the page in the end) either order is okay

(41) Algorithms of Finding the inverse of a matrix A

① put A in its row echelon form & record the steps

② put A further in its REDUCED row echelon form & record the steps (check if you get the identity matrix!)

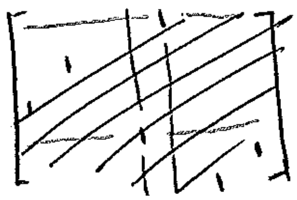
③ Perform all steps in ① & ② in SEQUENCE (in order) to the identity matrix I, the answer is the inverse of A.

Why does this work?? (Do example # 31 from the "exaples" page)

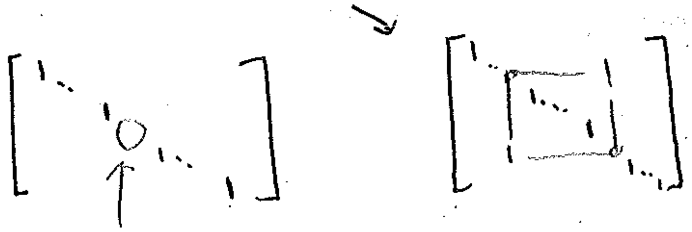
(42) e.g of ~~ele~~ elementary matrices

Type I. $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & \circ & & 1 \end{bmatrix}$

single non-zero entry off-diagonal

Type II,  result of swapping two rows of I

~~or: exactly one "1" in each row & column~~

Type III. 

exactly one entry not ~~necessarily~~ equal to 1 on the diagonal

(Do example # 32, # 33 from the "examples" page)

(43) Remark: All elementary matrices are INVERTIBLE!

(i.e. their inverses exists)

(* 2/25)

(44) Remark. The following are equivalent.

- A square matrix A is invertible
- The columns of A are linearly independent
- The eq. $A\vec{x} = \vec{0}$ has a unique solution
- * The row echelon form of A has no zero rows. (i.e. every row has a pivot)

(45) Def For a two-by-two matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

its ~~the~~ determinant is $ad - bc$. denoted as $\det A$

Fact. A is invertible if and only if $\det A \neq 0$.

(46) Fact $(AB)^{-1} = B^{-1}A^{-1}$

Proof. $(AB)\cdot(B^{-1}A^{-1}) = A \cdot I \cdot A^{-1} = A \cdot A^{-1} = I$

(47) Def The determinant of a ~~to~~ three-by-three matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$ is

$$\det A = a \cdot \det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b \cdot \det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right)$$
$$+ c \cdot \det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$

~~(48) Geometric Interpretation of determinants~~

(48) Fact: if $f: \mathbb{R}^2 \to \mathbb{R}^2$

$f(\vec{v}) = A\vec{v}$

then ~~A~~ $\det A$ is the rate at which f compresses area. / stretches

Remark. if $\det A = 0$,

then the image of f degenerates

to a single line

(49) Def for a matrix A

we use A_{ij} for the matrix that is the result

of deleting the i^{th} row & j^{th} colum in A

(50) Def. The determinant of a 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is: $\det A = a_{11} A_{11} - a_{12} A_{12} + a_{13} A_{13}$

(51) Fact A is invertible if and only if ~~A~~ $\det A \neq 0$

(i.e. A is NOT invertible if $\det A = 0$)

~~48.5~~ 49 Properties of the determinant.

$$\det(AB) = (\det A)\cdot(\det B)$$

50 Effect of row operations on the determinant.

Scaling by k $\longrightarrow$ det gets multiplied by k

interchange $\longrightarrow$ negate det.

replace $\longrightarrow$ no change

$(R\cdot c + R \curvearrowright R)$

(* updated 3/4)

NSC 228 Thur 11 am - 12:20 pm 3/14

~~Recall~~

37 def for $A = \begin{bmatrix} a_{11} \cdots & a_{1n} \\ a_{21} \cdots & a_{2n} \\ \vdots & \\ a_{n1} \cdots & a_{nn} \end{bmatrix}$ $A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & & \vdots \\ \vdots & \vdots & & \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$

Fact $\det A = \det A^T$.

Consequence if A is invertible then A^T is also invertible
-- NOT -- ---- NOT ----

38 Since columns in A = rows in A^T
& vice versa

A: square matrix if ~~rows~~ col in A are linearly indep
then col in A^T are linear indep.
then rows in A are also linearly indep.

Recall:
if $\vec{v}_1, \cdots, \vec{v}_n$ are linearly independent
span $\langle \vec{v}_1, \cdots, \vec{v}_n \rangle$ is defined to have dimension n.

39 Fact. if $\vec{w}$ is a linear combo of $\vec{v}_1, \cdots, \vec{v}_n$
then
span $\langle \vec{v}_1, \cdots, \vec{v}_n, \vec{w} \rangle$ = span $\langle \vec{v}_1, \cdots, \vec{v}_n \rangle$ = span $\langle \vec{v}_1, \cdots, \vec{v}_{n-1}, \vec{w} \rangle$

40 Fact. # of maximally indep. vectors in $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$
is the # of pivots in the matrix $A = [\vec{v}_1 \ \vec{v}_2 \cdots \vec{v}_n]$
also known as the dimension of ~~the~~ Span $\langle \vec{v}_1, \cdots, \vec{v}_n \rangle$
or "~~row~~ column Span" of A

Def Call this Rank A. and Col A : the column span of A

41 ~~De~~ Fact. Row reductions /elementary row operations do not change the Rank of a matrix

42 $A = [\vec{v}_1, \cdots, \vec{v}_n]$

The null space of A is the solution set of

$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = 0$ denoted null A

Nullity of A = dimension of null space

43 Row A = The space spanned by all rows in A.

"row space of A" do e.g. 52

44 Rank A = dim Col A

= dim Row A

= # of pivots in A

45 Recall: if $\vec{v}_1, \cdots, \vec{v}_n$ are linearly independent

$\{\vec{v}_1, \cdots, \vec{v}_n\}$ is called

a basis of $\text{span}\langle \vec{v}_1, \cdots, \vec{v}_n \rangle$

do e.g. 57 do e.g. 53, 54

~~46 F.~~ 46 Fact the vectors which span

Row A & Null A

form a basis of $\mathbb{R}^n$ where

n = # of columns in A.

(47) Consequence of (46)

$$n = \dim \text{Row } A + \dim \text{Null } A$$
$$= \text{Rank } A + \text{Nullity of } A$$

(48) If $\vec{w}$ is a linear combination of $\vec{v_1}, \dots, \vec{v_n}$

i.e. $\vec{w} = x_1\vec{v_1} + x_2\vec{v_2} + \dots + x_n\vec{v_n}$

then ~~the~~ $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is called the vector of <u>coordinates</u> of $\vec{w}$ under the ORDERED basis $\vec{v_1}, \dots, \vec{v_n}$.

denoted as $[\vec{w}]_{\{\vec{v_1}, \dots, \vec{v_n}\}}$

do e.g ~~54~~ 55, 56

~~(49) (Geometric Inte~~

<u>Remark</u>

(49) The coordinates of a given vector depend on the <u>basis</u>

do e.g. 58

(50) let $\{\vec{v_1}, \dots, \vec{v_n}\}$, $\{\vec{w_1}, \dots, \vec{w_n}\}$ be two bases of the same vector space

~~(50)~~ <u>Def</u>. The ~~matrix of~~ "change of coordinates" matrix for $\{\vec{v_1}, \dots, \vec{v_n}\}$ to $\{\vec{w_1}, \dots, \vec{w_n}\}$

is $\left[[\vec{v_1}]_{\{\vec{w_1}, \dots, \vec{w_n}\}} \; [\vec{v_2}]_{\{\vec{w_1}, \dots, \vec{w_n}\}} \; \cdots \; [\vec{v_n}]_{\{\vec{w_1}, \dots, \vec{w_n}\}} \right]$

~~denoted as~~ $P_{\{\cdots\}}$

do e.g. 59

(51) ~~$B = \{\vec{w}_1, \ldots, \vec{w}_n\}$~~

Fact P_1 change of coordinate matrix fr $\{\vec{w}_1, \ldots, \vec{w}_n\}$ to $\{\vec{v}_1, \ldots, \vec{v}_n\}$

P_2 — — — — $\{\vec{v}_1, \ldots, \vec{v}_n\}$ to $\{\vec{w}_1, \ldots, \vec{w}_n\}$

$$P_1 \cdot P_2 = I$$

do e.g. (60)

(52) Fact let P: change of coordinates matrix from $\{\vec{w}_1, \ldots, \vec{w}_n\}$ to $\{\vec{v}_1, \ldots, \vec{v}_n\}$

then given a vector $\vec{x}$

$$[\vec{x}]_{\{\vec{v}_1, \ldots, \vec{v}_n\}} = P \cdot [\vec{x}]_{\{\vec{w}_1, \ldots, \vec{w}_n\}}$$

e.g. (61)

* updated 3/11

(56) Def. $P\cancel{B}P^{-1}$ is called conjugation by P

(57) Th In general.

P: change of coordinate matrix from $\{\vec{w}_1, \vec{w}_2\}$ to $\{\vec{u}_1, \vec{u}_2\}$

A: def matrix for f under $\{\vec{w}_1, \vec{w}_2\}$

$\{\vec{u}_1, \vec{u}_2\}$

B

$$B = P^{-1}AP$$

(58) Eigenvectors and eigenvalue

Def. $f: \mathbb{R}^2 \to \mathbb{R}^2$

if $f(\vec{v}) = \lambda\vec{v}$ for λ: a real number

then $\vec{v}$ is an eigenvector for f of eigenvalue λ

(59) Th. if λ is an eigenvalue for f. A: defining matrix for f

Then λ is a root of

$$\det(A - \lambda I) = 0.$$

(60) Th eigenvalues are INDEPENDENT of the choice of basis. whereas the coordinates of the eigenvector do depend on the choice of basis

52 Recall if A: 2×2 matrix

53 $f: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(\vec{v}) = A\vec{v}$

We call A: the ~~associa~~ matrix associated to f

(defining matrix for)

Also. if $f: \mathbb{R}^2 \to \mathbb{R}^2$ linear

$$A = [\, f(\vec{e}_1) \quad f(\vec{e}_2) \,]$$

54 Def If $\{\vec{v}_1, \vec{v}_2\}$ is a basis of $\mathbb{R}^2$.

$f: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(\vec{v})_{\{\vec{v}_1, \vec{v}_2\}} = A\, \vec{v}_{\{\vec{v}_1, \vec{v}_2\}}$$

Then A: defining matrix for f under the basis $\{\vec{v}_1, \vec{v}_2\}$.

Notice $\quad \vec{v}_1{}_{\{\vec{v}_1, \vec{v}_2\}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{v}_2{}_{\{\vec{v}_1, \vec{v}_2\}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Hence $\quad A = [\, f(\vec{v}_1)_{\{\vec{v}_1, \vec{v}_2\}} \quad f(\vec{v}_2)_{\{\vec{v}_1, \vec{v}_2\}} \,]$

55 Th. if P: change of coordinate matrix from $\{\vec{v}_1, \vec{v}_2\}$ to $\{\vec{e}_1, \vec{e}_2\}$

A: def. matrix for A under $\{\vec{e}_1, \vec{e}_2\}$

B: - - - - - - $\{\vec{v}_1, \vec{v}_2\}$

Then $\quad A = PBP^{-1}$

or. $\quad P^{-1}AP = B$

(b1) Algorithm of finding eigen~~vectors~~ values

Solve the equation

$$\det(A-\lambda I)=0$$

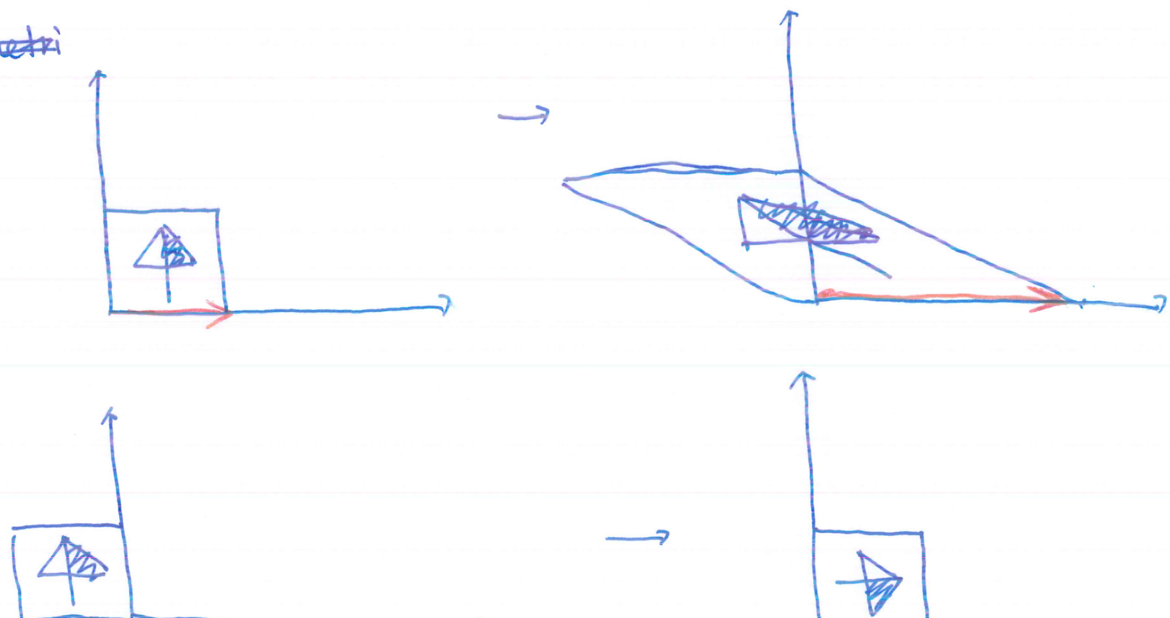
Algorithm of finding eigenvector(s) of eigenvalue λ.

Solve the SLE

$$(A-\lambda I)\vec{x}=\vec{0}$$

(b2) Remark. eigenvectors are well-defined up to a scalar multiple

(b3) ~~Geometri~~



No eigenvectors

(64) Th. $\det(B) = \det(PBP^{-1})$

Proof
$$\det(PBP^{-1}) = \det(P)\det(B)\det(P^{-1})$$
$$= \det(P)\det(P)^{-1}\det(B)$$
$$= \det(B)$$

(* updated 3/25)

Th The eigenvectors of the same eigenvalue form a vector space called the eigenspace of that eigenvalue

Def A matrix is diagonal if all nonzero entries are on the diagonal.

~~Def.~~ Th. ~~The~~ All eigenvalues of a diagonal matrix are the diagonal entries.

"Proof".

$$\begin{bmatrix} 7 & & \\ & 4 & \\ & & 1 \end{bmatrix} - \begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{bmatrix} = \begin{bmatrix} 7-\lambda & & \\ & 4-\lambda & \\ & & 1-\lambda \end{bmatrix}$$

$$\det \begin{bmatrix} 7-\lambda & & \\ & 4-\lambda & \\ & & 1-\lambda \end{bmatrix} = (7-\lambda)(4-\lambda)(1-\lambda)$$

Th ① $\det \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = \lambda_1 \lambda_2 \cdots \lambda_n$

② $\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mu_1 & & & \\ & \lambda_2\mu_2 & & \\ & & \ddots & \\ & & & \lambda_n\mu_n \end{bmatrix}$

$$\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}^a = \begin{bmatrix} \lambda_1^a & & & \\ & \lambda_2^a & & \\ & & \ddots & \\ & & & \lambda_n^a \end{bmatrix}$$

Def. B ~~A~~ is diagonalizable if

$$B = ADA^{-1}$$

for some diagonal matrix D

~~Def.~~

~~Fact. If $\vec{v}$ is the steady sto~~ ~~p~~ state

Def. The <u>state vector</u> of a stochastic event. is

$$\vec{v} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

where p_i is the probability of each event.

Def. The steady-state vector is ~~such~~ a vector $\vec{w}$

such that

$$M . \vec{w} = \vec{w}$$

~~i.e.~~ where M is the stochastic matrix

~~after d Days.~~

Fact. The state vector tomorrow is

$$\vec{u} = M\vec{v}$$

where $\vec{v}$ is the state vector today

after d days $\vec{u} = M^d \vec{v}$

A. $n \times n$ matrix

Def. If the (linearly-independent) vectors in each eigenspace of A. collectively form n vectors.

then they form a basis of $\mathbb{R}^n$

Fact eigenvectors of ~~linearly indi~~ different eigenspaces are linearly indep.

(* updated 4/4)

(The diagonalization Theorem)

Let A be an $n \times n$ square matrix

if A has n linearly independent vectors

$\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}$

Then A is diagonalizable

$$A = PDP^{-1}$$

where $P = [\vec{v_1} \ \vec{v_2} \ \cdots \ \vec{v_n}]$

i.e. the columns in P are eigenvectors of A

$$\& \quad D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

The ~~eio~~ entries in D are eigenvalues of each eigenvector $\vec{v_1}, \cdots, \vec{v_n}$ in that order.

Remark

Diagonalization corresponds to picking a set of eigenvectors of a linear map, use them as a basis, and ~~change~~ & compare the two defining matrices under the two bases.

Def. If there are n linearly independent eigenvectors of an $n \times n$ matrix A. We call them an eigenbasis of $\mathbb{R}^n$.

Definitions

(correction)

if P: change of coordinate matrix from $\{\vec{v}_1, \cdots, \vec{v}_n\}$ to $\{\vec{w}_1, \cdots, \vec{w}_n\}$

A: defining matrix for f under $\{\vec{v}_1, \cdots, \vec{v}_n\}$

B : — — — — — — $\{\vec{w}_1, \cdots, \vec{w}_n\}$

Then $B = PAP^{-1}$

D36.

Defintions

Observe: in Markov chains.

on one hand, the stable-state vector / vector of eigenvalue 1 is "the limit" of $A^k \vec{v_0}$ as $k \to \infty$ for any initial state-vector $\vec{v_0}$.

Q. In general, can eigen~~val~~vectors be obtained by computing $A^k \vec{v_0}$ for large enough k ?

A Yes, but only for one type of eigenvectors.

~~Def~~ **Eg** A: 3×3 $\quad A\vec{v_1} = -2\vec{v_1} \quad A\vec{v_2} = \vec{v_2} \quad A\vec{v_3} = \frac{1}{2}\vec{v_3}$

i.e. $\vec{v_1}$ eigenvector of eigenvalue -2.

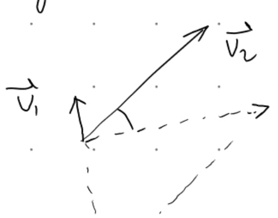
then $\vec{v_1}$ is called a dominant eigen-vector

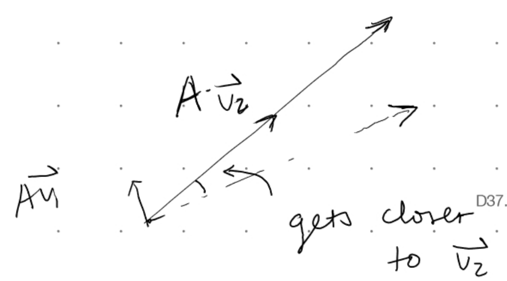
Def In general a dominant eigenvector is a vector whose eigenvalue has the ~~with~~ (unique) largest absolute value among all eigenvalues.

Th For any initial vector $\vec{v_0}$

The limit of $A^k \vec{v_0}$ as $k \to \infty$ is a dominant vecto after "rescaling" such that the largest entry is 1

(The geometric intuition)





Definitions

Def The graphical form of a vector $\vec{v} \in \mathbb{R}^2$ $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$

Is an arrow from the origin to the point with coordinates (x, y)

Def $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \in \mathbb{R}^2$ $\vec{w} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$

$\vec{v} \cdot \vec{w} = x_1 x_2 + y_1 y_2 \in \mathbb{R}$

Is call the <u>dot product</u> between $\vec{v}$ and $\vec{w}$.

~~Fact~~ **Def.** In general. $\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ $\vec{w} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$

$\vec{v} \cdot \vec{w} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$

is called the dot product between $\vec{v}$ & $\vec{w}$.

Def $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$

$\|\vec{v}\| = \sqrt{x^2 + y^2}$ is called the <u>magnitude</u> of $\vec{v}$

In general, $\vec{v} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

$\|\vec{v}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

1HR4-8

Definitions

Fact / Th $\vec{v} \in \mathbb{R}^2$ or $\vec{v} \in \mathbb{R}^3$

$\|\vec{v}\|$ is the length of the arrow in the graphical form of $\vec{v}$.

Fact / Th $\vec{v}, \vec{w} \in \mathbb{R}^2$ or $\vec{v}, \vec{w} \in \mathbb{R}^3$

Let θ be the angle between $\vec{v}$ & $\vec{w}$

then $\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\theta$

Def The projection of $\vec{v}$ along the direction of $\vec{w}$, is

$$\frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} \cdot \vec{w}$$

Remark The projection along $\vec{w}$ is a vector in the same direction as $\vec{w}$.

Def. $\vec{v}, \vec{w} \in \mathbb{R}^n$

$\vec{v}$ and $\vec{w}$ are said to be orthogonal to each other if $\vec{v} \cdot \vec{w} = 0$

Fact If $\vec{v}, \vec{w} \in \mathbb{R}^2$ or $\vec{v}, \vec{w} \in \mathbb{R}^3$

$\vec{v}, \vec{w}$ are orthogonal if and only if $\theta = 0$

<u>Fact</u>. $\vec{v}$ and $\vec{w}$ are orthogonal

if and only if the projection of $\vec{v}$ along $\vec{w}$ is 0

<u>Def</u> $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}\}$ is an ~~bas~~ orthogonal ~~orthonormal~~ basis of

$\mathbb{R}^n$ if they are pair-wise orthogonal and

form a basis of $\mathbb{R}^n$.

<u>Observation</u> given $\{\vec{v}, \vec{w}\}$ two linearly independent vectors

$\vec{v}$ ~~$\vec{w}$~~ $-$ (the projection of $\vec{v}$ along $\vec{w}$)

is always orthogonal to $\vec{w}$

& if $\vec{v}, \vec{w}$ are linearly dependent

$\vec{v}$ $-$ (the projection of ~~$\vec{w}$~~ $\vec{v}$ along $\vec{w}$) $= \vec{0}$

Theorem (The Gram-Schmidt Process)

Given $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}\}$ a basis of $\mathbb{R}^n$

Let $\{\vec{w_1}, \vec{w_2}, \dots, \vec{w_n}\}$ be defined as follows

$\vec{w_1} = \vec{v_1}$

$\vec{w_2} = \vec{v_2}$ $-$ (the projection of $\vec{v_2}$ along $\vec{w_1}$)

$$= \vec{v_2} - \frac{\vec{v_2} \cdot \vec{w_1}}{\|\vec{w_1}\|^2} \vec{w_1}$$

$$\vec{w_3} = \vec{v_3} - (\text{the projection of } \vec{v_3} \text{ along } \vec{w_1})$$

$$- (\text{the projection of } \vec{v_3} \text{ along } \vec{w_2})$$

$$= \vec{v_3} - \frac{\vec{v_3} \cdot \vec{w_2}}{\|\vec{w_2}\|^2} \vec{w_2} - \frac{\vec{v_3} \cdot \vec{w_1}}{\|\vec{w_1}\|^2} \cdot \vec{w_1}$$

$\vdots$

$\vec{w_n} = \vec{v_n}$ - sum of the projection of $\vec{v_n}$ along previous $\vec{w_i}$'s

Then $\{\vec{w_1}, \dots, \vec{w_n}\}$ form an orthogonal basis of $\mathbb{R}^n$

Properties of dot product

(Theorem) It is linear in each entry, i.e.

$$(c\vec{v_1}) \cdot \vec{v_2} = \vec{v_1} \cdot (c\vec{v_2}) = c(\vec{v_1} \cdot \vec{v_2})$$

$$(\vec{v_1} + \vec{v_2}) \cdot \vec{v_3} = \vec{v_1} \cdot \vec{v_3} + \vec{v_2} \cdot \vec{v_3}$$

$$\vec{v_1} \cdot (\vec{v_2} + \vec{v_3}) = \vec{v_1} \cdot \vec{v_2} + \vec{v_1} \cdot \vec{v_3}$$

Q What other operation(s) satisfy this properties?

Def. If "$\star$" is an operation between n-dimensional vectors, whose output is a real number

i.e. for $\vec{v_1} \in \mathbb{R}^n$ $\vec{v_2} \in \mathbb{R}^n$

$\vec{v_1} \star \vec{v_2}$ is a real number and "☆" satisfies the linear property.

then "☆" is called a quadratic form on $\mathbb{R}^n$

Definitions

Theorem All Quadratic Forms are in the following form:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \star \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} =$$ A function that is a linear combination of $x_i y_j$

(Defining matrix for a quadratic form)

Theorem If $\star$ is a Quadratic form on $\mathbb{R}^n$ then there exists a unique $n \times n$ matrix A such that

$$\vec{v} \star \vec{w} = \vec{v}^T A \vec{w}$$

Moreover, the (i,j)-th entry in A is the coefficient of the term $x_i y_j$ in the quadratic form.

~~Theorem~~ T

~~Def. The notion of "orthogonal set" still applies in the conte when the dot product gets replaced by the quadratic form~~

<u>Th</u> The Gram-Schmidt process still produces an orthogonal set when the operation is ~~any~~ quadratic form

Definitions

<u>Remark</u> The notion of "orthogonality" isn't well-defined when we don't specify the order of the the two inputs.

<u>Def</u>. A real quadratic form is called an <u>inner-product</u> if its defining matrix is symmetric

<u>Def</u>. A matrix X is called <u>symmetric</u> if

$$A = A^T$$

<u>Theorem</u> If $\star$ is an inner-product, then

$$\vec{v} \star \vec{w} = \vec{w} \star \vec{v}$$

<u>Proof</u>.

$$\vec{v} \star \vec{w} = \vec{v}^T A \vec{w}$$

$$\vec{w} \star \vec{v} = \vec{w}^T A \vec{v}$$

equal

$$(\vec{w} \star \vec{v})^T = \vec{v}^T A^T \vec{w}$$

<u>Theorem</u> If $\star$ is an inner product
$\vec{v} \star \vec{v} \geq 0$ and equality is achieved only when $\vec{v} = \vec{0}$

Def. When $\star$ is an inner product

then define

$$\|\vec{v}\| = \sqrt{\vec{v} \star \vec{v}}$$

this inner product

the norm of $\vec{v}$ with respect to $\star$

Recall in Gram-Schmidt

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \circ \vec{w}_1}{\|\vec{w}_1\|^2} \cdot \vec{w}_1$$

Th The Gram-Schmidt ~~still~~ process still produces an orthogonal set with respect to an inner product.

Th. With respect to an inner product, an orthogonal set is ALWAYS linearly independent.

Definitions

Def. An <u>LU Factorization</u> of $\underset{n\times n}{\underset{\uparrow}{A}}$ is the expression

$$A = LU$$

where L is an $n\times n$ lower triangular matrix with 1's on the diagonal and U is an upper triangular matrix.

Def. A matrix is <u>upper-triangular</u> if it looks like

$$\begin{bmatrix} * & * & * & \cdots & * \\ 0 & * & * & & \\ 0 & 0 & * & \ddots & \\ 0 & \vdots & & \ddots & * \\ 0 & 0 & \cdots & 0 & * \end{bmatrix}$$

& <u>lower-triangular</u> if it looks like

$$\begin{bmatrix} * & 0 & \cdots & & 0 \\ * & * & 0 & \ddots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ * & * & \cdots & & * \end{bmatrix}$$

Definitions

Th If $A = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ * & & & d_n \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 1/d_1 & & & 0 \\ & 1/d_2 & & \\ & & \ddots & \\ * & & & 1/d_n \end{bmatrix}$$

Similarly, the same is true for upper-Δ matrices

Algorithm for LU-factorization

(1) Row reduce the matrix A into its row echelon form, using only "replacement operation"

(2) record the sequence of row operations as multiplication by elementary matrices.