(18). Describe all solutions (the solution set) of the following homogeneous SLE: (rep via its augmented matrix)

$$\begin{bmatrix} 3 & 4 & 1 & 0 \\ 1 & 2 & -1 & 0 \end{bmatrix}$$

(19) Describe the solution sets of the ~~following~~ homogeneous SLE with following augmented matrix

$$\begin{bmatrix} -1 & 2 & 3 & 4 & 0 \\ 5 & 1 & 0 & -1 & 0 \end{bmatrix}$$

(20) Determine if the following ~~SLE (rep using an~~ vectors are linearly independent.

1) $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

2) $\vec{v}_1 = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} 7 \\ -4 \\ 3 \end{bmatrix}$ $\quad \vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

3) $\vec{v}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\quad \vec{v}_3 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

(* updated 2/7)

21 4) $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\vec{e_4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

22 Set up the SLE which balances the following chemical equation.

$$C_3H_8 + O_2 \longrightarrow CO_2 + H_2O$$

23 Set up the SLE which solves the distribution of the economy among the three sectors:

x_1 (percentage) in Coal

x_2 in Electric

x_3 in Steel

with the following distribution table

From / to	Coal	Electric	Steel
Coal	.2	.1	.2
Electric	.4	.3	.5
Steel	.4	.6	.3

(24) Based on the following (incomplete) information, explain why the given map is NOT linear

1) f

$\mathbb{R}^2$	$\mathbb{R}^3$
$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}3\\4\\2\end{bmatrix}$
$\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}-1\\1\\3\end{bmatrix}$
$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}2\\5\\4\end{bmatrix}$

2) $f\left(\begin{bmatrix}2\\4\end{bmatrix}\right) = \begin{bmatrix}1\\3\end{bmatrix}, \quad f\left(\begin{bmatrix}3\\6\end{bmatrix}\right) = \begin{bmatrix}\frac{3}{2}\\4\end{bmatrix}$

3) $f\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\7\end{bmatrix}, \quad f\left(\begin{bmatrix}0\\1\\6\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix},$

$$f\left(\begin{bmatrix}1\\2\\0\end{bmatrix}\right) = \begin{bmatrix}\cancel{6}7\\7\end{bmatrix}$$

(25.) Compute the image of the linear map f on the following vectors, given the (complete) information about $f: \mathbb{R}^2 \to \mathbb{R}^4$ (or g)

1) $f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}3\\-1\\4\\7\end{bmatrix}, \quad f\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\1\\7\end{bmatrix}$

compute $f\left(\begin{bmatrix}2\\3\end{bmatrix}\right)$

~~25~~

25. cont.

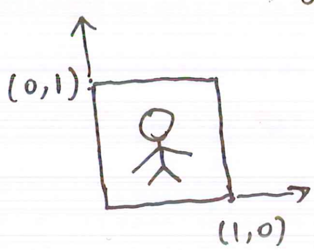
2) $f\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ 0 \\ -1 \end{bmatrix}$ $\quad$ $f\left(\begin{bmatrix} 1 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

$f\left(\begin{bmatrix} 1 \\ 18 \end{bmatrix}\right) = \,?$ $\quad$ $f\left(\begin{bmatrix} -3 \\ -10 \end{bmatrix}\right) = \,?$

26 If

$$f : \mathbb{R}^2 \to \mathbb{R}^2$$
$$\vec{v} \mapsto \begin{bmatrix} 2 & \\ & 1 \end{bmatrix} \cdot \vec{v}$$

Determine the image of the following picture



27 Do Problem 26 For the maps that are left multiplication by the following matrices

① $\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$ $\quad$ ② $\begin{bmatrix} 1 & 2 \\ & 1 \end{bmatrix}$ $\quad$ ③ $\begin{bmatrix} 1 & \\ & \end{bmatrix}$

28 Determine the matrix associated to the following linear map

1) $f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $\quad$ $f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

28

2)

$$f\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$f\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(* updated 2/18)

examples

29) Turn the following matrix into their reduced row echelon form

① $\begin{bmatrix} 2 & 7 & 3 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ ② $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & -2 \end{bmatrix}$

30) Find the inverse of the following matrix, (use the row reduction algorithm)

① $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ ② $\begin{bmatrix} 1 & 2 & 3 \\ & 4 & 6 \\ & & -2 \end{bmatrix}$ (from #29)

31) 1) Let $A = \begin{bmatrix} 3 & 7 & 4 \\ 1 & 2 & -1 \\ 0 & -1 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & 4 \\ 0 & -1 & 4 \end{bmatrix}$

identify the elementary row operation, and come up with a matrix C such that

$$A = C \cdot B$$

2) Do the same for $A = \begin{bmatrix} 1 & 7 & 4 \\ 0 & -1 & 3 \\ 2 & 7 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 21 & 12 \\ 0 & -1 & 3 \\ 2 & 7 & 4 \end{bmatrix}$

3) Do the same for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}$

(32) Find the inverses of the following matrix

1) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ 2) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ 3) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

(either use row reduction algorithm, or
use the elementary row operation that "undoes" it.
~~the~~ ~~preference~~ or both depending on
time constraint)

(33) For the following matrix, write it as a product of
elementary matrices and the identity matrix

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

Also, write its inverse as a product of elementary
matrices and the identity matrix

(* 2/25)

(34) Determine if the following matrix has an inverse

$$\begin{bmatrix} 1 & -2 & -3 \\ 7 & 0 & 7 \\ -2 & -1 & \cancel{0}\ -4 \end{bmatrix}$$

(35) Determine if the following vectors are linearly independent.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 7 \\ -2 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix}$$

(36) Determine if $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ has a unique solution.

where

$$A = \begin{bmatrix} 0 & -1 & 2 \\ 5 & -6 & 1 \\ 5 & -9 & 7 \end{bmatrix}$$

~~(37) Given $\vec{v} = \begin{bmatrix} 1 \\ 7 \\ -2 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}$~~

~~Compute $\vec{v} + 2\vec{w}$,~~

~~and determine if $\begin{bmatrix} 1 & 7 & -2 \\ -2 & 0 & -1 \end{bmatrix}$~~

(37) Find the inverse of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

& give the condition on the parameters so that A is invertible

(38) ~~Find the inverse~~

Find the determinant of

$$A = \begin{bmatrix} 3 & 7 \\ 4 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 7 \\ 1 & \frac{7}{4} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 7 & 5 \\ 2 & 4 & 3 \\ -1 & 0 & -2 \end{bmatrix} \qquad D = \begin{bmatrix} -1 & 2 & 7 \\ 2 & 0 & -3 \\ 0 & 4 & 11 \end{bmatrix}$$

and determine if they are invertible

(39) Use the determinants, determine whether the following vectors are linearly independent.

$$\vec{v}_1 = \begin{bmatrix} \frac{1}{2} \\ -1 \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ -\frac{1}{3} \end{bmatrix} \qquad \vec{v}_3 = \cancel{\begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{3} \end{bmatrix}} \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -\frac{2}{3} \end{bmatrix}$$

(37) (40) Find the area of the parallelogram that is the image of the unit square under the map

$$f: \mathbb{R}^2 \to \mathbb{R}^2$$

$$f(\vec{v}) = \begin{bmatrix} \sqrt{3} & 2 \\ 1 & 0 \end{bmatrix} \vec{v}$$

and $g: \mathbb{R}^2 \to \mathbb{R}^2$

$$g(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \vec{v}$$

(43) (42) ~~Notation for~~ A_i

Compute the determinant of the following matrix

$$A = \begin{bmatrix} -4 & 7 & 11 \\ 3 & 1 & -3 \\ -1/4 & 2 & 2 \end{bmatrix}$$

on the other hand, determine if A is invertible (or if the columns are linearly independent)

(* updated 3/4)

(44)

(41) Compute $\det(AB)$

and $\det(A) \cdot \det(B)$

for

$$A = \begin{bmatrix} 1 & 7 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 0 \\ -3 & 8 \end{bmatrix}$$

(42) Compute the determinants before/after the following row operations

$$\begin{bmatrix} -2 & 7 \\ -4 & 1 \end{bmatrix} \xrightarrow{R_2 \cdot \pi} \begin{bmatrix} -2 & 7 \\ -4\pi & \pi \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} -4 & 1 \\ -2 & 7 \end{bmatrix}$$

$$\xrightarrow{R_1 \cdot 314 + R_2 \rightsquigarrow R_2} \begin{bmatrix} -2 & 7 \\ -2 \cdot 314 - 4 & 7 \cdot 314 + 1 \end{bmatrix}$$

(45) For

$$A = \begin{bmatrix} 3 & 7 & -1 \\ 2 & -2 & 4 \\ 6 & 8 & 0 \end{bmatrix}$$

Compute $\det A^T$

$\det A^{-1}$

Verify $A \cdot A^{-1} = I$

$$A = \begin{bmatrix} -4 & 7 & 11 \\ 3 & 1 & -3 \\ 0 & 2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-8} \begin{bmatrix} 8 & 8 & -32 \\ -6 & -8 & 21 \\ 6 & 8 & -25 \end{bmatrix}$$

(46) ~~Find the dimens a maximum set of linearly independent vectors~~

Given $\vec{c} = 3\vec{a} + 2\vec{b}$

and $\vec{d} = 7\vec{a} - \vec{b} + 4\vec{c}$

write $\vec{d}$ as a linear combination of $\vec{a}, \vec{b}$

also, write $\vec{d}$ - - - - $\vec{a}, \vec{c}$

(47) Find a maximal collection of linearly independent vectors from

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} -5 \\ -3 \\ 18 \end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix} 7 \\ -1 \\ -7 \end{bmatrix} \quad \vec{v}_5 = \begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix}$$

and conclude $\dim \operatorname{span} \langle \vec{v}_1, \dots, \vec{v}_5 \rangle$

(48) In (45), Find Rank A, Rank A^{-1}, Rank A^T

(49) Write the solutions set of (47) as the span of (linearly independent vectors) vectors.

and conclude dim null A

(50) Find Rank A.

$$A = \begin{bmatrix} 7 & -11 & 1 & & 1 \\ -1 & 5 & 1 & & 1 \\ 4 & -11 & -1 & & -1 \\ 0 & 6 & 2 & & 2 \end{bmatrix}$$

(51) Based on the information, write R_3 as a linear combo of R_1, R_2
on row operations

$$[\quad] \xrightarrow{R_1 \cdot 3} [\quad] \xrightarrow[R_2 + R]{R_2 = \quad R_1 \cdot 4} [\quad] \xrightarrow[+R_3]{R_3 = 3 \cdot R_2} [\,0 \quad 0 \quad 0\,]$$

EXAMPLES

52 Determine the dimension of the ~~solution~~ null space of A

(b) ~~(4a)~~ & the rank of A. Also find a basis of Row A & Null A.

$$A = \begin{bmatrix} 1 & 7 & 2 & 0 \\ 2 & -1 & 3 & -2 \\ -4 & 0 & 1 & 9 \end{bmatrix}$$

53 Determine if the ~~spanning vectors of~~ four vectors which span Row A & Null A are linearly independent.

54 ~~Write~~ Write $\vec{v} = \begin{bmatrix} 2 \\ 7 \\ -1 \\ 3 \end{bmatrix}$ as a linear combination of the four vectors in 53 using matrix inversion.

55 Find the coordinates of $\vec{v}$

1) under the standard basis

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{e_4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

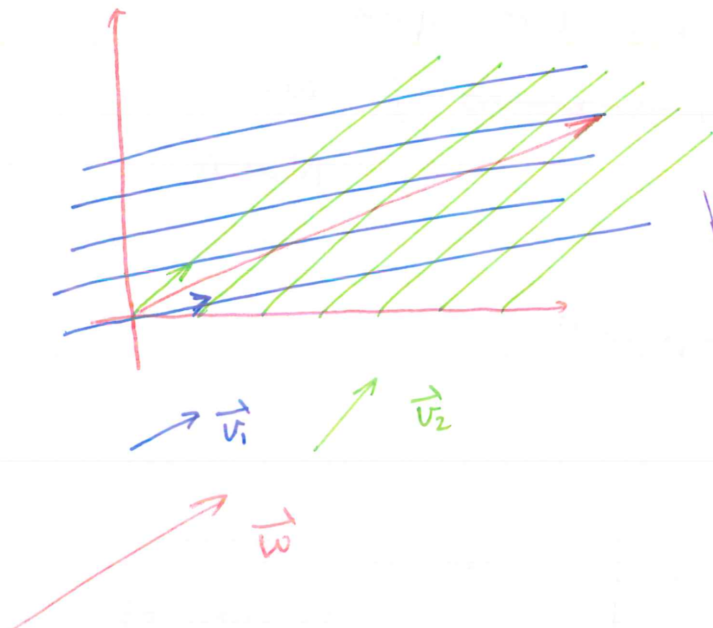
2) and the ordered basis in 53

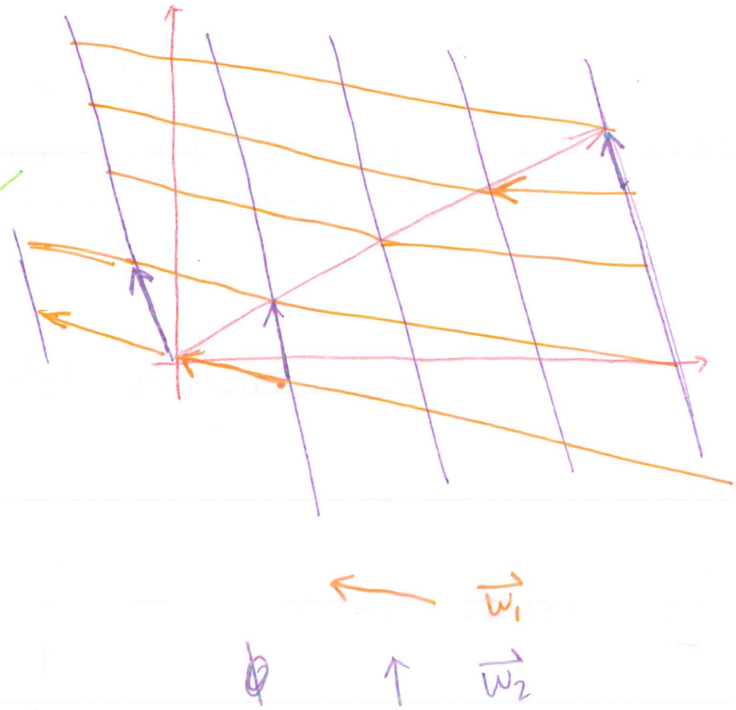
(56) (Geometric Interpretation of Coordinates)

~~Write~~ Find the coordinates of $\vec{w}$ in $\mathbb{R}^2$ under the basis

1) $\vec{v_1}, \vec{v_2}$

2) $\vec{w_1}, \vec{w_2}$





(57) Determine if each of the following form a basis of $\mathbb{R}^2$

1) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

2) $\begin{bmatrix} 2 \\ 7 \end{bmatrix}$ $\begin{bmatrix} -6 \\ -21 \end{bmatrix}$

3) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 6 \end{bmatrix}$

4) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

5) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

(58) given

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

use matrix inversion. find the coordinates of

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

under the ~~standard~~ basis $\{\vec{v}_1, \vec{v}_2\}$

(59) ~~Find the change of~~ Using the vectors in (58)

Find the change of coordinates matrix

1) from ~~$\vec{e}$~~, $\{\vec{v}_1, \vec{v}_2\}$ to $\{\vec{e}_1, \vec{e}_2\}$

2) from $\{\vec{e}_1, \vec{e}_2\}$ to $\{\vec{v}_1, \vec{v}_2\}$

Check that the two matrices are inverses of each other

(60) let

$$\vec{u} = \begin{bmatrix} \cancel{-9} \\ 8 \end{bmatrix}$$

$$\vec{w}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \qquad \vec{w}_2 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

a) If $[\vec{u}]_{\{\vec{w}_1, \vec{w}_2\}} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, find $\vec{u}_{\{\vec{e}_1, \vec{e}_2\}}$

On the other hand, ~~let~~ P: ~~ma~~

Compute P: the change of coordinates matrix from $\{\vec{w}_1, \vec{w}_2\}$ to $\{\vec{e}_1, \vec{e}_2\}$

and compute $P \cdot [\vec{w}]_{\{\vec{w}_1, \vec{w}_2\}}$

b) if $[\vec{s}]_{\{\vec{e}_1, \vec{e}_2\}} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

from scratch (without using matrix multiplication)

compute $[\vec{s}]_{\{\vec{w}_1, \vec{w}_2\}}$

~~and~~ On the other hand,

compute $P^{-1} \cdot [\vec{s}]_{\{\vec{e}_1, \vec{e}_2\}}$

(b1) Given ~~$\vec{w}_1, \vec{w}_2$~~ $\vec{w}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ $\vec{w}_2 = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$

given $[\vec{x}]_{\{\vec{w}_1, \vec{w}_2\}} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$

Compute $[\vec{x}]_{\{\vec{v}_1, \vec{v}_2\}}$

* updated 3/11

(62) Under the basis $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$f: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad f(\vec{w}) = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \vec{w}$

(using $\{\vec{v}_1, \vec{v}_2\}$-coordinates)

~~find~~ if i.e. $f(\vec{w})_{\{\vec{v}_1, \vec{v}_2\}} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \vec{w}_{\{\vec{v}_1, \vec{v}_2\}}$.

if $\vec{w}_1{}_{\{\vec{e}_1, \vec{e}_2\}} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$

$\vec{w}_2{}_{\{\vec{v}_1, \vec{v}_2\}} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

find $f(\vec{w}_1)_{\{\vec{v}_1, \vec{v}_2\}}$

and $f(\vec{w}_2)_{\{\vec{e}_1, \vec{e}_2\}}$

(63) Let $\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

find: the change-of-coordinate matrix from $\{\vec{v}_1, \vec{v}_2\}$ to $\{\vec{e}_1, \vec{e}_2\}$

Also if the defining matrix for f is

$$A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \text{ under } \{\vec{e}_1, \vec{e}_2\}.$$

(from scratch, without using the Th) find the defining matrix B for f under $\{\vec{v}_1, \vec{v}_2\}$

Also compute $P^{-1}BP$.

(64) let $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$\vec{w}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $\vec{w}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

find the change-of-coordinate matrix from $\{\vec{v}_1, \vec{v}_2\}$ to $\{\vec{w}_1, \vec{w}_2\}$

in addition, if the defining matrix for f is

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ under $\{\vec{v}_1, \vec{v}_2\}$.

find the defining matrix under $\{\vec{w}_1, \vec{w}_2\}$.

& compute $f(\vec{u})_{\{\vec{w}_1, \vec{w}_2\}}$ for $\vec{u}_{\{\vec{w}_1, \vec{w}_2\}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(65) e.g.

Find the eigen value for the following eigen vectors

$A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $A\vec{v} = 5\vec{v}$

$A = \begin{bmatrix} -7 & -1 & -4 \\ -1 & -5 & 0 \\ -1 & -4 & 6 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$ $A\vec{v} = -8\vec{v}$

$A = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $A\vec{v} = 0 \cdot \vec{v}$

(66) Find an eigenvector(~~s~~) of ~~a~~ eigenvalue 1 for the following matrices

$$A = \begin{bmatrix} 5 & -3 \\ -4 & 4 \end{bmatrix}$$

~~B =~~

(68) For the above matrices.

(~~67~~) Compute $\det(A - I)$

$\det(B - 2I)$

(67) Find an eigenvector(~~s~~) of eigenvalue 2 for the following matrix

$$B = \begin{bmatrix} 3 & -4 & -3 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

(69) Find all eigenvalues for the following matrices.

$$A = \begin{bmatrix} 11 & -4 \\ 30 & -11 \end{bmatrix} \quad C = \begin{bmatrix} -7 & 4 \\ -25 & 11 \end{bmatrix} \quad B =$$

$$B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

~~70~~ Find ~~all~~ ~~eigenvalue / eigenvector~~ for the following ~~matrices~~.

70 If $\vec{w}_1 = 2\vec{v}_1 - 3\vec{v}_2$

$\vec{w}_2 = -\vec{v}_1 + \vec{v}_2$

The defining matrix for f under $\{\vec{v}_1, \vec{v}_2\}$ is $\begin{bmatrix} 7 & -1 \\ 0 & 2 \end{bmatrix}$

find . . — — — — $\{\vec{w}_1, \vec{w}_2\}$

(* updated 4/1)

(71) ~~Find all eigenvalues / eigenvectors~~ for the following matrix

each of the

Describe ~~the~~ eigenspaces ~~linearly indep.~~ ~~of~~ as the span of one or more vectors

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 8 & -4 \\ 9 & -4 \end{bmatrix}$$

(72) <u>Application of eigenvalues</u>.

Given the following exchange table.

	Steel	Coal	Electricity
Steel	0.7	0.4	0.2
Coal	0.2	0.4	0.3
Electricity	0.1	0.3	0.5

if ~~or~~ x_1, x_2, x_3 are the proportions of each sector in a equilibrium

find the matrix for which

$$\vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

is an eigenvector of

& state the eigenvalue

73 Markov Chains (4.9)

Suppose the ~~wthe~~ weather in Buffalo follows the pattern below

(Sunny. Cloudy. Rain. Snow)

When it is sunny today, the probability of four types of ~~weather~~ weather tomorrow is as follows.

Sunny	0.5	Cloudy	0.1	Rain	0.2	Snow	0.2

(Cloudy today)

0.3	0.4	0.1	0.2

Stochastic matrix →

(Rain)

0.1	0.3	0.3	0.3

(Snow)

0.2	0.1	0.1	0.6

If $\vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ is the ~~state~~ Steady-state of the probability of weather.

find the matix for which $\vec{v}$ is an eigenvalue of

& State the eigenvalue

If the probability vector today is

$\vec{v_0} = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \end{bmatrix}$ (a) find the prob. vector tomorrow & in Ten days. (answer can be left unsimplified)

State the Stochastic matrix. & find the Steady-State vector.

(b) (set up the equations). state its eigenvalue

(74) If

$$A = \begin{bmatrix}2 & 7\\ 4 & 1\end{bmatrix}\begin{bmatrix}1 & \\ & -2\end{bmatrix}\begin{bmatrix}2 & 7\\ 4 & 1\end{bmatrix}^{-1}$$

Find A^{100}.

(75) ~~e.g. $A = [\quad]$ has eigenba~~

~~eigenvector~~

(75) If A has the following eigenspace decomposition.

1 - eigenspace span $\left\langle \begin{bmatrix}1\\2\\7\end{bmatrix}, \begin{bmatrix}-1\\0\\-1\end{bmatrix} \right\rangle$

-1 - eigenspace span $\left\langle \begin{bmatrix}0\\0\\2\end{bmatrix} \right\rangle$

~~find~~ ~~State~~ let $f: \mathbb{R}^2 \to \mathbb{R}^2$ $f(\vec{v}) = A\vec{v}$

state ~~the~~ an eig-basis of f and

state the <u>defining</u> matrix under that basis.

76

	Apple	~~Orag~~ Orange
Apple	0.3	0.9
Orange	0.7	0.1

Find the stable-state vector for this stochastic process.

(* updated 4/4)

Examples

(77) Diagonalize the following matrix

$$A = \begin{bmatrix} -8 & 10 \\ -5 & 7 \end{bmatrix}$$

(78) Determine if the following matrix is diagonalizab

$$A = \begin{bmatrix} -6 & 7 \\ -7 & 8 \end{bmatrix}$$

(79) Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a basis of $\mathbb{R}^3$

let P: change-of-coordinate matrix

from $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ to $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

a) if $P = \begin{bmatrix} 2 & 7 & 3 \\ 1 & 4 & 0 \\ -1 & 0 & 7 \end{bmatrix}$, state $\vec{v}_1, \vec{v}_2, \vec{v}_3$

b) if $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are eigenvectors of $\cancel{A}$ $\cancel{B}$ f

with eigenvalue $1, -2, 7$

~~and if B is the defining matrix~~ for

a linear map $f: \mathbb{R}^3 \to \mathbb{R}^3$

(79) cont.)

Find the defining matrix of f under the basis $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$

(3) Find the defining matrix of f under the basis $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$

80 Let $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$

Compute the vector after every iriteration & its difference with $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ after rescaling.

(or $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$)

Examples

(82) Compute $\vec{v} \cdot \vec{w}$ and $\|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\theta$

for

a) $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$

b) $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$

(83) Given

$\vec{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix}$ $\vec{u} = \begin{bmatrix} 8 \\ 0 \\ 14 \end{bmatrix}$

find:

a) the projection of $\vec{v}$ along $\vec{w}$

b) the projection of $\vec{w}$ along $\vec{v}$

c) the projection of $\vec{v}$ along $\vec{u}$.

(X) d) if $\vec{t}$ is the answer in a)

check if $\vec{w}$ and $(\vec{v} - \vec{t})$ are orthogonal

e) the projection of $\vec{v}$ along $\vec{v}$

D6.

84 Check if each of the following set is orthogonal & linearly independent.

a) $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} -3 \\ 6 \end{bmatrix} \right\}$

b) $\left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

c) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 10 \\ 4 \end{bmatrix} \right\}$

d) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

e) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(85) let $\vec{u} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

a) compute $\vec{t}$ = the projection of $\vec{u}$ along $\vec{w}$

b) check if $\vec{w}$ and $(\vec{u} - \vec{t})$ are orthogonal

c) Construct an orthogonal basis of $\mathbb{R}^2$ from $\{\vec{u}, \vec{w}\}$

(86) Given

$$\vec{a} = \begin{bmatrix} 1 \\ 7 \\ -2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Construct an orthogonal basis from $\vec{a}, \vec{b}, \vec{c}$

(87) Determine if the following operation on $\mathbb{R}^2 \times \mathbb{R}^2$ is a quadratic form.

a) $\vec{v_1} * \vec{v_2} = \begin{bmatrix} \text{the largest entry in } \vec{v_1} \\ \end{bmatrix}$

b) $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} * \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cancel{x_1 + x_2} \\ \cancel{y_1 + y_2} \end{bmatrix}$ $x_1 + x_2 + y_1 + y_2$

c) $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \star \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1 y_2 *$

d) $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \star \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1 y_1$

Examples

(88) State whether each of the following is a quadratic form

a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \star \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 x_2 + y_1 y_2$

b) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \star \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 y_2$

c) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \star \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = x_1 y_2^2 + x_3 y_1$

d) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \star \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = x_1 y_3 + 2 x_2 y_1$

e) $\vec{v} \star \vec{w} = \vec{v}^T \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{w}$

($\vec{v}, \vec{w}$ in $\mathbb{R}^2$)

f) $\vec{v} \star \vec{w} = \vec{v}^T \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & 0 & -1 \end{bmatrix} \vec{w}$

($\vec{v}, \vec{w}$ in $\mathbb{R}^3$)

Examples

(89) Identify the defining matrix for the following quadratic form.

a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} * \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 3x_1y_1 - 2x_1y_2 + x_2y_1$

b) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} * \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = x_1y_3 - 2x_2y_1 + 3x_1y_2 - y_2x_1 - y_1x_3 + 4x_2y_2$

c) The dot product

(90) With respect to the quadratic form defined by $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, check if each of the set is orthogonal

a) $\{\vec{e_1}, \vec{e_2}\}$

b) $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with respect to itself

~~Also check if~~ $\cancel{\vec{e_1}}$

c) $\left\{ \vec{v_1} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

examples

(q1) with respect to an inner product defined by

$$\begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}$$

Check if ~~each of~~ the following is an orthogonal set

a) $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad \vec{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

b)

(q2) with respect to an inner product defined by

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

construct an orthogonal set from

a) $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \vec{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

b) $\vec{v_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad \vec{v_2} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Examples.

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$$A = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ & 1 & -1 \\ & & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 0 \\ 6 & 5 & -1 \\ 0 & -1 & -1 \end{bmatrix}$$

find $\det A$

find a solution to $A\vec{x} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$

Write A^{-1} as

(upper Δ) $\cdot$ (lower Δ)

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$$A = \begin{bmatrix} 3 & 1 & 0 \\ & 2 & -4 \\ & & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & & \\ 2 & -1 & \\ 3 & -1 & 4 \end{bmatrix}$$

find A^{-1}, B^{-1}

(94) Find an LU decomposition of

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 8 & 3 \end{bmatrix}$$