Presenting hyperoctahedral Schur algebras

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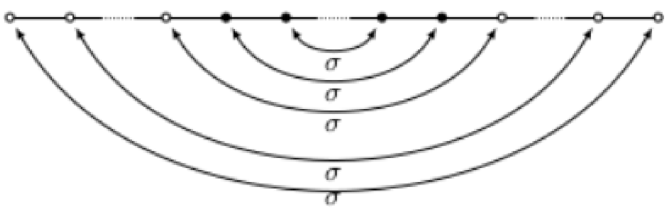
The fixed point subalgebra

$N = 2n + 1$, $n \in \mathbb{Z}_{\geq 0}$

Γ: Dynkin diagram of Type A_{N-1}

$\sigma \in \mathrm{Aut}(\Gamma)$



ω: involution of $\mathfrak{gl}_N(\mathbb{C})$

a Lie algebra automorphism:

$$\begin{aligned} \theta := \omega \circ \sigma : \mathfrak{gl}_N(\mathbb{C}) &\to \mathfrak{gl}_N(\mathbb{C}) \\ e_i &\mapsto f_{N-i} \\ f_i &\mapsto e_{N-i} \\ h_i &\mapsto h_{N+1-i} \end{aligned}$$

$\mathfrak{g}^\theta = \mathfrak{gl}_N(\mathbb{C})^\theta$: the subalgebra of $\mathfrak{g} = \mathfrak{gl}_N(\mathbb{C})$ fixed by θ

A presentation of $\mathfrak{g}^\theta$

(Letzter: quantum case)

(Li-Z.) $U(\mathfrak{g}^\theta)$ has a presentation with generators

$$\begin{aligned} \mathbf{e}_i &= e_i + f_{N-i} & 1 \le i \le n \\ \mathbf{f}_i &= f_i + e_{N-i} & 1 \le i \le n \\ \mathbf{d}_i &= h_i + h_{N+1-i} & 1 \le i \le n+1 \end{aligned}$$

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$$\mathbf{d}_i = h_i + h_{N+1-i} \qquad 1 \le i \le n+1$$

subject to relations

$$\mathbf{e}_i^2\mathbf{f}_i - \mathbf{e}_i\mathbf{f}_i\mathbf{e}_i + \mathbf{f}_i\mathbf{e}_i^2 = -4\mathbf{e}_i$$
$$\mathbf{f}_i^2\mathbf{e}_i - \mathbf{f}_i\mathbf{e}_i\mathbf{f}_i + \mathbf{e}_i\mathbf{f}_i^2 = -4\mathbf{f}_i$$
$$[\mathbf{d}_i, \mathbf{e}_j] = (\delta_{i,j} - \delta_{i,j+1})\mathbf{e}_j \quad (1 \le i, j \le n)$$
$$[\mathbf{d}_{n+1}, \mathbf{e}_j] = -2\delta_{n,j}\mathbf{e}_j \quad (1 \le j \le n)$$

and more

Quantum Type B Duality

$U_q(\mathfrak{g}^\theta)$: algebra generatedy by $\mathbf{e}_1, \ldots, \mathbf{d}_{n+1}$, subject to quantized relations

$U_q(\mathfrak{g}^\theta) \hookrightarrow U_q(\mathfrak{gl}_N)$: quantum group

V: natural $U_q(\mathfrak{gl}_N)$-module

$H_q(B)$: Hecke algebra of Type B

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(Bao-Kujawa-Li-Wang)

$$U_q(\mathfrak{g}^\theta) \twoheadrightarrow \mathrm{End}_{H_q(B)}(V^{\otimes d})$$

(Shoji-Sakamoto)

$$U_q(\mathfrak{gl}_{n+1}(\mathbb{C})) \otimes U_q(\mathfrak{gl}_n(\mathbb{C})) \twoheadrightarrow \mathrm{End}_{H_q(B)}(V^{\otimes d})$$

The action of the Weyl group

Let $B_d = S_d \ltimes (\mathbb{Z}_2)^d$, $s = (\mathrm{id}, (\overline{1}, \overline{0}, \ldots, \overline{0})) \in B_d$

B_d: generated by S_d and s.

$B_d \curvearrowright V^{\otimes d}$:

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$B_d \curvearrowright V^{\otimes d}$:

1) (Green)
$V = \langle e_1, \dots, e_n, e_{n+1}, e_{\overline{n}}, \dots, e_{\overline{1}} \rangle$,

$s.e_{i_1} \otimes \cdots \otimes e_{i_d} = e_{\overline{i_1}} \otimes \cdots \otimes e_{i_d}$.

$(\overline{n+1} = n+1)$

$B_d \curvearrowright V^{\otimes d}$:

2) (Hu-Stoll, Mazorchuk-Stroppel)
$V = \langle w_1, \ldots, w_{n+1}, w_{\overline{1}}, \ldots, w_{\overline{n}} \rangle$

$$V_1 = \langle w_1, \ldots, w_{n+1} \rangle, \qquad V_{-1} = \langle w_{\overline{1}}, \ldots, w_{\overline{n}} \rangle$$

$s \curvearrowright V_1 \otimes V^{\otimes d-1}$ via 1

$s \curvearrowright V_{-1} \otimes V^{\otimes d-1}$ via -1

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1) equivalent to 2) under

$$w_i = e_i + e_{\overline{i}} \quad 1 \leq i \leq n+1$$
$$w_i = e_i - e_{\overline{i}} \quad 1 \leq i \leq n$$

An explicit isomorphism

Let E_i, F_i, H_i: Chevalley generators in $\mathfrak{gl}_{n+1}(\mathbb{C})$

$E_{\bar{i}}, F_{\bar{i}}, H_{\bar{i}}$: Chevalley generators in $\mathfrak{gl}_n(\mathbb{C})$

Theorem (Li-Z.)

There is an isomorphism of algebras

$$\begin{aligned}
\phi : U(\mathfrak{g}^\theta) &\to U(\mathfrak{gl}_{n+1}(\mathbb{C}) \oplus \mathfrak{gl}_n(\mathbb{C})) \\
\mathbf{e}_i &\mapsto E_i + E_{\bar{i}} \quad (1 \le i \le n-1) \\
\mathbf{f}_i &\mapsto F_i + F_{\bar{i}} \quad (1 \le i \le n-1) \\
\mathbf{h}_i &\mapsto H_i + H_{\bar{i}} \quad (1 \le i \le n) \\
\mathbf{e}_n &\mapsto 2E_n \\
\mathbf{f}_n &\mapsto F_n \\
\mathbf{h}_{n+1} &\mapsto 2H_{n+1} + 1
\end{aligned}$$

Root vectors in $\mathfrak{gl}_{n+1}(\mathbb{C})$

$\epsilon_i = H_i^*$, $(1 \leq i \leq n+1)$.

$X_{\epsilon_i - \epsilon_{i+1}} = E_i \in \mathfrak{gl}_{n+1}(\mathbb{C})$

$$
\begin{aligned}
(i < j) \qquad X_{\epsilon_i - \epsilon_j} &= X_{(\epsilon_i - \epsilon_{j-1}) + (\epsilon_{j-1} - \epsilon_j)} \\
&= [X_{\epsilon_i - \epsilon_{j-1}}, X_{\epsilon_{j-1} - \epsilon_j}] \\
&= [X_{(\epsilon_i - \epsilon_{j-2}) + (\epsilon_{j-2} - \epsilon_{j-1})}] \\
&= [[X_{\epsilon_i - \epsilon_{j-2}}, X_{\epsilon_{j-2} - \epsilon_{j-1}}], X_{\epsilon_{j-1} - \epsilon_j}] \\
&\vdots
\end{aligned}
$$

$X_{\epsilon_i - \epsilon_j} = E_{ij} \in \mathfrak{gl}_{n+1}(\mathbb{C})$

Similarly, identify $\epsilon_i = H_{\bar{i}}^*$, define $Y_{\epsilon_i - \epsilon_j} \in \mathfrak{gl}_n(\mathbb{C})$.

Root vectors in $\mathfrak{g}^\theta$

Recall $h_1, \ldots, h_{2n+1} \in \mathfrak{gl}_{2n+1}(\mathbb{C})$

$\mu_i = h_i^*$, $(1 \leq i \leq 2n+1)$.

Φ_{2n+1}^+: positive roots in $\mathfrak{gl}_{2n+1}(\mathbb{C})$

Define $Z_\alpha \in \mathfrak{g}^\theta$ for all $\alpha \in \Phi_{2n+1}^+$:

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Φ^+_{2n+1}: positive roots in $\mathfrak{gl}_{2n+1}(\mathbb{C})$

Define $Z_\alpha \in \mathfrak{g}^\theta$ for all $\alpha \in \Phi^+_{2n+1}$:

On simple roots

$$Z_{\mu_1-\mu_2} = \mathbf{e}_1, \quad \ldots \quad Z_{\mu_n-\mu_{n+1}} = \mathbf{e}_n$$
$$Z_{\mu_{n+1}-\mu_{n+2}} = \mathbf{f}_n, \quad \ldots \quad Z_{\mu_{2n}-\mu_{2n+1}} = \mathbf{f}_1$$

In general,

$$\begin{aligned} Z_{\mu_i-\mu_j} &= Z_{(\mu_i-\mu_{j-1})+(\mu_{j-1}-\mu_j)} \\ &= [Z_{\mu_i-\mu_{j-1}}, Z_{\mu_{j-1}-\mu_j}] \\ &\vdots \end{aligned}$$

Identify $\epsilon_i = \mathbf{d}_i^*$, $(1 \leq i \leq n+1)$

$\Phi^\theta = \{\epsilon_i - \epsilon_j \mid 1 \leq i, j \leq n+1, i \neq j\} \simeq \Phi_{n+1}$

Define $p : \Phi_{2n+1}^+ \to \Phi^\theta$

on a simple root α,

Z_α: weight vector for $\mathbf{d}_i$ of weight $p(\alpha)$

$$
\begin{aligned}
& p : \Phi^+_{2n+1} \to \Phi^\theta \\
\text{“}\mathbf{e}_1\text{”} \qquad & \mu_1 - \mu_2 \mapsto \epsilon_1 - \epsilon_2 \\
& \vdots \\
\text{“}\mathbf{e}_n\text{”} \qquad & \mu_n - \mu_{n+1} \mapsto \epsilon_n - \epsilon_{n+1} \\
\text{“}\mathbf{f}_n\text{”} \qquad & \mu_{n+1} - \mu_{n+2} \mapsto -\epsilon_n + \epsilon_{n+1} \\
& \vdots \\
\text{“}\mathbf{f}_1\text{”} \qquad & \mu_{2n} - \mu_{2n+1} \mapsto -\epsilon_1 + \epsilon_2
\end{aligned}
$$

In general: extend linearly.

Weight spaces in $\mathfrak{g}^\theta$

Lemma (Li-Z.)

Let $\alpha \in \Phi^+_{2n+1}$, then X_α is a weight vector for $\mathbf{d}_1, \ldots, \mathbf{d}_{n+1}$ of weight $s(\alpha)$.

Weight spaces in $\mathfrak{g}^\theta$

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Let $\alpha \in \Phi^+_{2n+1}$, then X_α is a weight vector for $\mathbf{d}_1, \ldots, \mathbf{d}_{n+1}$ of weight $s(\alpha)$.

For $\alpha \in \Phi^\theta \cup \{0\}$, let

$$\mathfrak{g}^\theta_\alpha = \{x \in \mathfrak{g}^\theta \mid [\mathbf{d}_i, x] = \alpha(\mathbf{d}_i)x, \quad \forall 1 \le i \le n+1\}$$

In particular, let $\mathfrak{h} \subset \mathfrak{gl}_{n+1}(\mathbb{C}) \oplus \mathfrak{gl}_n(\mathbb{C})$, $\dim \mathfrak{h} = 2n+1$

$$s^{-1}(0) = \{\mu_i - \mu_{2n+2-i} \mid 1 \le i \le n\}$$
$$h_i := Z_{\mu_i - \mu_{2n+2-i}} \in \mathfrak{g}^\theta_0$$

Also

$$s^{-1}(\epsilon_i - \epsilon_j) = \{\mu_i - \mu_j, \mu_i - \mu_{2n+2-j}\} \qquad (1 \leq i < j \leq n)$$

$$W_{\epsilon_i - \epsilon_j} := \frac{1}{2} Z_{\mu_i - \mu_{2n+2-j}} \in \mathfrak{g}^{\theta}_{\epsilon_i - \epsilon_j}$$

$$W'_{\epsilon_i - \epsilon_j} := Z_{\mu_i - \mu_j} \in \mathfrak{g}^{\theta}_{\epsilon_i - \epsilon_j}$$

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And

$$s^{-1}(\epsilon_i - \epsilon_{n+1}) = \{\mu_i - \mu_{n+1}\} \qquad (1 \le i \le n)$$
$$W_{\epsilon_i - \epsilon_{n+1}} := \frac{1}{2} Z_{\mu_i - \mu_{n+1}} \in \mathfrak{g}^{\theta}_{\epsilon_i - \epsilon_{n+1}}$$

and similarly define $W_{-\epsilon_i + \epsilon_j}$, $W'_{-\epsilon_i + \epsilon_j}$

Explicit inverses

Recall $\phi : \mathfrak{g}^\theta \to \mathfrak{gl}_{n+1}(\mathbb{C}) \oplus \mathfrak{gl}_n(\mathbb{C})$

Let α: root in $\mathfrak{gl}_{n+1}(\mathbb{C})$

β: root in $\mathfrak{gl}_n(\mathbb{C})$

Theorem (Li-Z.)

$$\begin{aligned}\phi(h_i) &= H_i \qquad\qquad (1 \leq i \leq n)\\ \phi(W_\alpha) &= X_\alpha\\ \phi(W'_\beta) &= X_\beta + Y_\beta\end{aligned}$$

Recall $\phi(\mathbf{d}_i) = H_i + H_{\bar{i}}$ $(1 \le i \le n)$.

Define $h_{\bar{i}} = \mathbf{d}_i - h_i$. Then $\phi(h_{\bar{i}}) = H_{\bar{i}}$

Also let $h_{n+1} = \frac{1}{2}(\mathbf{d}_{n+1} - 1)$. Then $\phi(h_{n+1}) = H_{n+1}$

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Theorem (Kujawa-Z.)

The kernel of

$$U(\mathfrak{gl}_{n+1}(\mathbb{C}) \oplus \mathfrak{gl}_n(\mathbb{C})) \to \mathrm{End}_{\mathbb{C}B_d}(V^{\otimes d})$$

is the ideal generated by

$$\begin{aligned} H_1 + \cdots + H_{n+1} + H_{\bar{1}} + \cdots + H_{\bar{n}} &= d \\ H_i(H_i - 1) \cdots (H_i - d) &= 0 \\ H_{\bar{i}}(H_{\bar{i}} - 1) \cdots (H_{\bar{i}} - d) &= 0 \end{aligned}$$

A presentation of the Schur algebra

Theorem (Li-Z.)

The Schur algebra $\mathrm{End}_{\mathbb{C}B_d}(V^{\otimes d})$ is generated by

$$\mathbf{e}_1, \ldots, \mathbf{e}_n, \mathbf{f}_1, \ldots, \mathbf{f}_n, \mathbf{d}_1, \ldots, \mathbf{d}_{n+1}$$

subject to the $\mathfrak{g}^\theta$-relations, and

$$\mathbf{d}_1 + \cdots + \mathbf{d}_n + \frac{1}{2}(\mathbf{d}_{n+1} - 1) = d$$

$$h_i(h_i - 1) \cdots (h_i - d) = 0$$

$$h_{\bar{i}}(h_{\bar{i}} - 1) \cdots (h_{\bar{i}} - d) = 0$$

Theorem (Li-Z)

In $\mathrm{End}_{H_B}(V^{\otimes d})$, the following relations are satisfied

$$\mathbf{d}_1 + \cdots + \mathbf{d}_n + \frac{1}{2}(\mathbf{d}_{n+1} - 1) = d$$
$$\mathbf{d}_i(\mathbf{d}_i - 1)\cdots(\mathbf{d}_i - d) = 0$$
$$(\mathbf{d}_{n+1} - 1)(\mathbf{d}_{n+1} - 3)\cdots(\mathbf{d}_{n+1} - (2d+1)) = 0$$

Quantinzed version: Bao-Kujawa-Wang-Li

Theorem (Li-Z)

In $\mathrm{End}_{H_B}(V^{\otimes d})$, the following relations are satisfied

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Conjecture (Li-Z.)

The Schur algebra $\mathrm{End}_{\mathbb{C}B_d}(V^{\otimes d})$ is a quotient of $U(\mathfrak{g}^\theta)$ under the abvoe relations and

$$h_i(h_i - 1)\cdots(h_i - d) = 0 \qquad (1 \leq i \leq n)$$

Future work

- Cases when $N = 2n$
- A homomorphism $U_q(\mathfrak{g}^\theta) \to U_q(\mathfrak{gl}_{n+1}(\mathbb{C})) \otimes U_q(\mathfrak{gl}_n(\mathbb{C}))$?
- or the other direction?

Thank you!