① fix an enumeration $x_0, x_1, \dots$ of W
$x_i \le x_j \Rightarrow i \le j$

A ∇-filtration is
$$0 = B^0 \subseteq B^1 \subseteq \dots \subseteq B^k = B$$
s.t. $B^{i+1}/B^i \simeq R_{x_i}^{\oplus h'_{x_i}}$

$h'_{x_i} \in \mathbb{Z}[v, v^{-1}]$.

$h'_{x_i}(B) = h'_{x_i}$

$$ch_\nabla(B) = \sum_{x \in W} v^{\ell(x)} \overline{h'_x(B)} \, \delta_x$$
← standard basis

Remark
$$ch_\Delta(B \oplus B') = ch_\Delta(B) + ch_\Delta(B')$$
$$ch_\nabla(B \oplus B') = ch_\nabla(B) + ch_\nabla(B')$$
$$ch_\Delta(B(1)) = v \, ch_\Delta(B)$$
$$ch_\nabla(B(1)) = v^{-1} ch_\nabla(B)$$

(Soergel) Any Soergel bimod admits a unique ∇-filtration & is independent of the ~~filtrati~~ enumeration

$\mathbb{Z}[v, v^{-1}] \curvearrowright [\mathbb{S}Bim]_\oplus$

$v \curvearrowright$ by $B(1)$

Then ch_Δ is linear only $\mathbb{Z}[v, v^{-1}]$

Localization

Let Q: field of fractions for R

$$B_s \otimes_R Q = R \otimes_{R^s} R \otimes_R Q = R \otimes_{R^s} Q \hookrightarrow Q \otimes_{Q^s} Q$$

is surjective

Lemma

$$BS(\underline{w}) \otimes_R Q \simeq Q \otimes_{Q^s} Q \otimes_{Q^t} Q \otimes \dots \otimes_{Q^u} Q$$
$\shortparallel$
$(B_s \otimes B_t \otimes \dots \otimes B_u)$ as (R, Q)-bimod

Let $\mathbb{BS}Bim_Q$: smallest full subcat of Q-bimod containing all tensor products of $B_{s,Q} := B_s \otimes_R Q$

$\mathbb{S}Bim_Q$: …… containing all direct sums of direct summands of obj in $\mathbb{BS}Bim_Q$

$StdBim_Q$: …. containing all direct sums of Q_x $\forall x \in W$

Loc: $\mathbb{BS}Bim \longrightarrow \mathbb{BS}Bim_Q$

$- \otimes_R Q$ a monoidal functor

Lemma

$B_s \otimes_R Q \simeq Q_s \oplus Q$ as Q-bimod.

Proof:

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e.g.

$$0 \to R(-1) \to B_s \to R_s(1) \to 0$$
$$ch_\nabla(B_s) = v^0 \cdot \overline{v^{-1}} \cdot \delta_{id} + v^1 \cdot \overline{v} \cdot \delta_s$$
$$= v + \delta_s = b_s = ch_\Delta(B_s)$$

Remark. $ch_\nabla(B_x) = ch_\Delta(B_x)$ $\forall x \in W$

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cont) Proof:

Q: flat Q-mod

$- \otimes Q$ is exact

$$0 \to R_s \otimes Q \to B_s \otimes Q \xrightarrow{\mu} R \otimes Q \to 0$$

$B_s \otimes Q = Q \otimes_{Q^s} Q$, $\frac{1}{\alpha_s} \circ \mu$, $f \otimes g \overset{\mu}{\longmapsto} fg$

$$0 \to R \otimes Q \to B_s \otimes Q \xrightarrow[\mu_s]{} R_s \otimes Q \to 0$$
$R \otimes Q = Q$, $(1 \mapsto c_s)$

$$1 \mapsto \tfrac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s) \mapsto \frac{2\alpha_s}{2\alpha_s} = 1$$

$(Q_s \oplus Q) \otimes (Q_t \oplus Q)$

$Q_x \otimes Q_y \simeq Q_{xy}$

$$\mathbb{S}Bim_{\mathbb{Q}} \simeq StdBim_{\mathbb{Q}}$$

$$[StdBim_{\mathbb{Q}}]_{\oplus} \simeq \mathbb{Z}[W]$$

$$\begin{tikzcd}
\mathbb{S}Bim \arrow[r, "ch"] \arrow[d, "Loc"'] & H \arrow[d, "v=1"] \\
\mathbb{S}Bim_{\mathbb{Q}} \arrow[r] & \mathbb{Z}[W]
\end{tikzcd}$$

(Soergel's Cat Th.)

a) $\exists\ \mathbb{Z}[v,v^{-1}]$ - alg homomorphism

$$c:\ H \longrightarrow [SBim]_{\oplus}$$
$$b_s \mapsto [B_s]$$

b) $\exists$ 1-1 correspondence

$$W \longleftrightarrow \{\text{indecomposable obj in } SBim\}/\simeq, \quad (1)$$
$$w \longmapsto B_w$$

where B_w is uniquely characterized as the indecomposable summand of $BS(\underline{w})$. Moreover, all other summands in $BS(\underline{w})$ are various shifts of B_x for some $x<w$

c) $\exists$ inverse map to c

$$ch = ch_{\Delta}:\ [SBim]_{\oplus} \longrightarrow H$$

Hence both are isomorphism

Lemma can place "Moreover" by B_w has $R_w(-\ell(w))$ in its subquotients, and all other subquotients are shifts of R_x for $x<w$.

Proof. Deodhar's formula $\quad b_{\underline{w}} = \sum_{e \subseteq \underline{w}} v^{\text{defect}(e)} \delta_{\underline{w}^{e}}$

$$b_{\underline{s}} = b_s b_t \cdots b_u$$
$$= Ch(BS(\underline{w}))$$

defect$(\underline{w}) = 0$ $\quad$ no-DO

$$Ch(BS(\underline{w})) = \sum_{x\in W} v^{\ell(x)} h_x(BS(\underline{w}))\, \delta_x$$

$$1 = v^{\ell(w)} h_x(BS(\underline{w}))$$
$$h_x(BS(\underline{w})) = v^{-\ell(w)}$$

$\varphi \in \mathrm{Hom}(M,N)$ R-bim

$(r\cdot\varphi\cdot s)(m) = r\cdot\varphi(m)\cdot s$

$$\mathrm{Hom}^\bullet(M,N) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}(M,N(i)) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}(M(-i),N)$$

$$\mathrm{grk}\,\mathrm{Hom}^\bullet(M,N) = \sum_{i\in\mathbb{Z}} v^i \,\mathrm{rank}\,\mathrm{Hom}(M,N(i))$$

(Soergel's Cat. Th)

$$\mathrm{grk}\,\mathrm{Hom}(B,B') = (\mathrm{ch}(B), \mathrm{ch}(B'))$$
$$= \varepsilon(\omega(\mathrm{ch}(B))\cdot \mathrm{ch}(B'))$$

<u>Remark</u>: works w/ Sesquilinearity

$$\mathrm{grk}\,\mathrm{Hom}(B(1),B') = v^{-1}\,\mathrm{grk}\,\mathrm{Hom}(B,B')$$

(Soergel's conjecture)

$b_x \mapsto [B_x] \quad \forall x\in W$

Moreover, $h_x(B_y) = p_{x,y}$ the KL polynomials

$$(\, b_y = \sum p_{x,y}\, \delta_x \,)$$

<u>Eg</u> Prove Soergel's Th/conj for A_2