How to draw Soergel bimodules

Recall $B_s = R \otimes_{R^s} R(1)$ has a basis as a right R-mod (or left R-mod)

$$C_{id} = 1 \otimes 1 \qquad C_s = \tfrac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$$

deg -1 $\qquad$ deg 1

$BS(\underline{w}) = B_{s_1} \otimes \cdots \otimes B_{s_m}$ has a basis $\qquad \underline{e} \subseteq \underline{w}$ viewed as a 01 string

$\underline{e} = (e_1, e_2, \dots, e_m)$

$$C_{\underline{e}} = C_{s_1}^{e_1} \otimes_R C_{s_2}^{e_2} \otimes \cdots \otimes_R C_{s_m}^{e_m}$$

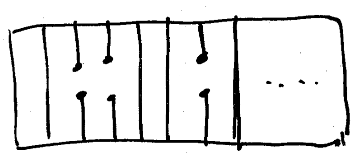
Basis: $\{C_{\underline{e}} \mid \underline{e} \subseteq \underline{w}\}$

define $\quad C_{bot} = C_{00\cdots0} = C_{id} \otimes \cdots \otimes C_{id} = 1 \otimes 1 \cdots \otimes 1 \qquad$ deg $-\ell(\underline{w})$

$(BS(\underline{w}) = (R \otimes_{R^{s_1}} R) \otimes (R \otimes_{R^{s_2}} R) \otimes \cdots$

$C_{top} = C_{01\cdots1} = C_{s_1} \otimes C_{s_2} \otimes \cdots \otimes C_{s_m} \qquad$ deg $\ell(\underline{w})$

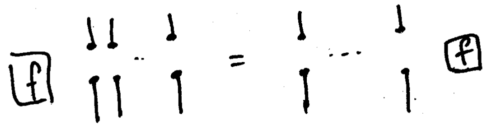
Define $\phi_{\underline{e}} \in \mathrm{End}^{\circ}(BS(\underline{w}))$



$e_i = 0$  $\qquad e_i = 1$ 

 $(1 \otimes 1) = C_s \qquad \phi_{\underline{e}}(C_{bot}) = C_{\underline{e}}$

Lemma $\qquad f \cdot C_{top} = C_{top} \cdot f$



Ring structure

$R \otimes_{R^s} R$: ring $\quad (f_1 \otimes f_2) \cdot (g_1 \otimes g_2) = f_1 g_1 \otimes f_2 g_2$

similar for $BS(\underline{w})$

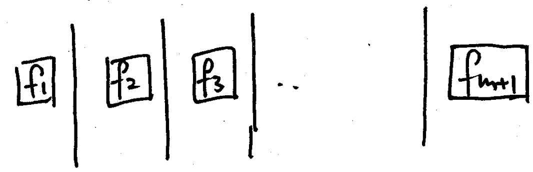
Not a graded ring $\quad 1 \otimes 1 \cdots \otimes 1$ has deg $-\ell(\underline{w})$)

$BS(\underline{w})(-\ell(\underline{w}))$ is a graded ring

$\exists$ homomorphism

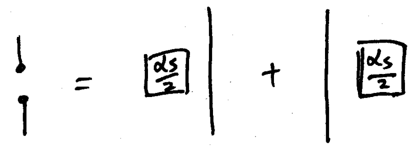
$$BS(\underline{w})(-\ell(\underline{w})) \longrightarrow \mathrm{End}^{\bullet}(BS(\underline{w}))$$

~~$f_1 \otimes f_2 \otimes$~~ · given by multiplication

$$f_1 \otimes f_2 \otimes \cdots \otimes f_{m+1} \longmapsto$$ 

denote the image by P.

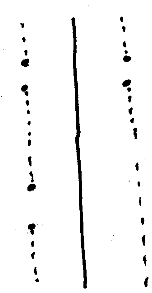
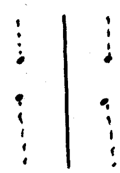
then $f, g \in P$. $\quad f(C_{bot}) \cdot g(C_{bot}) = f \circ g(C_{bot})$

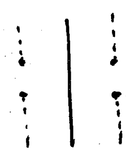
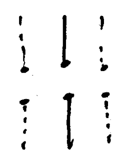
Also: $\phi_{\underline{e}} \in P$ 

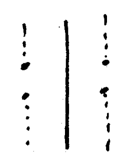
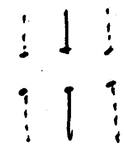
Prop. $\quad \phi_{\underline{e}}(C_{bot})\,\phi_{\underline{f}}(C_{bot}) = \phi_{\underline{e}} \circ \phi_{\underline{f}}(C_{bot})$

$\underline{e}, \underline{f} \subseteq \underline{w}$ $\qquad \overset{\shortparallel}{C_{\underline{e}} \cdot C_{\underline{f}}}$

Eg: $C_{101} \cdot C_{100}$

$\phi_{101} \circ \phi_{100} =$  $= \alpha_s$ 

$=$  $t(\alpha_s) +$  $\partial_t(\alpha_s)$

$=$  $t(\alpha_s) -$ 

$$C_{101} \cdot C_{100} = C_{101}\, t(\alpha_s) - C_{111}$$

$\langle C_{101}, C_{100} \rangle = -1$

· Global Intersection form

$$\mathrm{Tr}: \quad BS(\underline{w}) \to R$$

which picks out the coeff for C_{top}

define a symmetric, bilinear form

$$BS(\underline{w}) \times BS(\underline{w}) \longrightarrow R$$

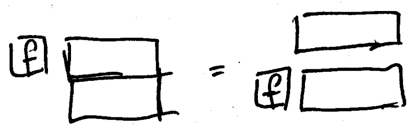
$$(a,b) \longmapsto \langle a,b\rangle = Tr(ab)$$

- symmetric

Prop. 1 $\deg\langle a,b\rangle = \deg(a) + \deg(b)$

2 $\langle fa, b\rangle = \langle a, fb\rangle \quad \forall f \in R$

3 $\langle af, b\rangle = \langle a, bf\rangle = \langle a,b\rangle f$



e.g. $C_s^2 = \alpha_s \cdot C_s$

$$C_{top} \cdot C_{top} = \alpha_{s_1} \overset{C_{s_m}}{\cancel{\alpha_s}} \otimes \cdots \quad \otimes \alpha_{s_m} C_{s_m}$$

$$= \prod_{i=1}^{m} \alpha_{s_i} \, C_{top} = C_{top} \prod_{i=1}^{m} \alpha_{s_i}$$

$$\langle C_{top}, C_{top}\rangle = \prod_{i=1}^{m} \alpha_{s_i}$$

Prop Up to ordering the col. lexicographically & the rows reverse lex...

$\underline{e}, \underline{f} \subseteq \underline{w}$ $\quad \underline{e} < \underline{f}$

$\langle \underline{e}, \underline{f}^{\circ}\rangle = 0$ $\quad$ $\underline{f}^{\circ}$ reversing all letters in $\underline{f}$, replace $0 \leftrightarrow 1$

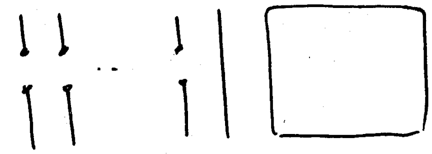
$\langle \underline{e}, \underline{e}^{\circ}\rangle = 1$

Hence, $\langle -,-\rangle$ is non-degenerate

Proof $\underline{e}_j = \underline{f}_j \quad j < i$

$\underline{e}_j$ and $\underline{f}_j^{\circ}$ are different

$\underline{e}_i = 0. \quad \underline{f}_i = 1$



$$\langle \underline{e}, \underline{f}^\circ \rangle = 0$$

$$\langle \underline{e}, \underline{e}^\circ \rangle = 1$$

$$\begin{array}{cc} & \underline{e} \\ \begin{array}{c} \\ \underline{e}^\circ \\ \underline{f}^\circ \\ \\ \end{array} & \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & 0 & \ddots \end{bmatrix} \end{array}$$

Basis for $BS(\underline{w})$

Recall: a basis for $\mathrm{End}^\bullet(BS(\underline{w}))$ is given by

$$\{ \overline{LL}_{\underline{w},\underline{f}} \circ LL_{\underline{w},\underline{e}} \mid \underline{w}^{\underline{f}} = \underline{w}^{\underline{e}}, \quad \underline{f}, \underline{e} \subset \underline{w} \}$$

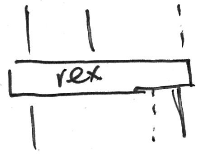
Goal: analyze $LL_{\underline{w},\underline{e}}(c_{bot})$

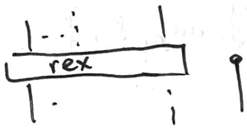
$\underline{e}$ consists of solely of up moves if the stroll for $\underline{e}$ only uses U0, U1

<u>Prop</u> $LL_{\underline{w},\underline{e}}(c_{bot}) = \begin{cases} c_{bot} \in BS(\underline{w}^{\underline{e}}) & \text{if } \underline{e} \text{ consists only of ups} \\ 0 & \text{o.w.} \end{cases}$

$c_{bot} \in BS(\underline{w})$

<u>Proof.</u>

U1 

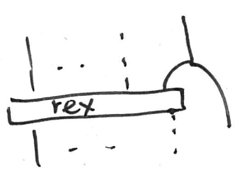
U0 

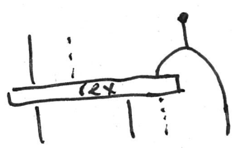
rex moves preserve c_{bot}:

 : $B_s B_t B_s \cdots \twoheadrightarrow B_{sts\cdots} \leftarrow B_t B_s B_t \cdots$

contains $R_w(-\ell(w))$ in the standard filtration

$1 \otimes 1 \cdots \otimes 1 \in R_w(-\ell(w))$

D0 

D1 

 : $f \otimes g \otimes h \mapsto \partial_t(g) f \otimes h = 0$

<u>Prop</u> if $x \leq w$

$\exists$ unique stroll $\underline{e} \subseteq \underline{w}$ consisting only of ups

s.t. $\underline{w}^{\underline{e}} = x$

call this the canonical exp of x, denoted as can_x

Any elt in $BS(\underline{w})$ can be viewed as

$$\phi(C_{bot}) \qquad (f_1 \otimes \cdots \otimes f_{m+1})$$

for some $\phi \in End^\circ(BS(\underline{w}))$

$$\phi = \sum_{\substack{\underline{e}, \underline{f} \subseteq \underline{w} \\ \underline{w}^{\underline{e}} = \underline{w}^{\underline{f}}}} \overline{LL}_{\underline{w}, \underline{f}} \circ LL_{\underline{w}, \underline{e}} \, g_{\underline{e}, \underline{f}}$$

Cor $$\phi(C_{bot}) = \sum_{\underline{f} \subseteq \underline{w}} \overline{LL}_{\underline{w}, \underline{f}} (C_{bot, \underline{w}^{\underline{f}}}) \, g_{can_{\underline{w}^{\underline{f}}}, \underline{f}}$$

$$C_{bot, \underline{w}^{\underline{f}}} \in BS(\underline{w}^{\underline{f}})$$

Th $\{ \overline{LL}_{\underline{w}, \underline{f}} (C_{bot, \underline{w}^{\underline{f}}}) \mid \underline{f} \subseteq \underline{w} \}$ form a basis of $BS(\underline{w})$

- Standard filtration

(∇-filtration) $\quad 0 = B_0 \subseteq B_1 \subseteq \cdots$

$$B_i / B_{i-1} = R_{x_i}^{h_i}$$

$$h_i \in \mathbb{N}[v, v^{-1}]$$

(fix a total order $x_1 < x_2 < \cdots$ which refines the Bruhat order)

Prop let B_i be the R-span of

$$\{ \overline{LL}_{\underline{w}, \underline{f}} (C_{bot}) \mid \underline{w}^{\underline{f}} = x_j, \ j \leq i \}$$

$$C_{bot} \in BS(\underline{w}^{\underline{f}})$$

Then $\{B_i\}$ forms a ∇-filtration for $BS(\underline{w})$