§ 9 Explicit Q-functions

Recall $\prod_{x \in X} \frac{1+x}{1-x} = \sum q_i(X)$ $\quad \langle q_i, q_j \rangle = \delta_{ij} \cdot 2$ if $i>0$

q_i: deg i piece $\quad \langle q_0, q_0 \rangle = 1$

Δ: subring of symmetric functions gen. by q_i (as a ring)

$q_\lambda = q_{\lambda_1} q_{\lambda_2} \cdots q_{\lambda_\ell}$

Q_λ : unique orthogonal basis of Δ s.t

Q_λ has q_λ as the leading term

Thm $\quad Q_\lambda = \beta_{\lambda_1} \circ \beta_{\lambda_2} \circ \cdots \circ \beta_{\lambda_\ell}(1)$

$\forall f \in \Delta$, define $f^\perp$ via $\langle f^\perp(g), h \rangle = \langle g, fh \rangle$

then $(f \cdot g)^\perp = g^\perp \cdot f^\perp = f^\perp \cdot g^\perp$

recall : $\Delta : \Delta \to \Delta \otimes \Delta \quad \Delta(q_n) = \sum_{i \in \mathbb{Z}} q_i \otimes q_{n-i}$

$$q_n^\perp(FG) = \sum_n q_i^\perp(F)\, q_{n-i}^\perp(G)$$

Also, $\langle q_n^\perp(q_i), f \rangle = \langle q_i, q_n f \rangle = \sum_{s=0}^{i} \langle q_s, q_n \rangle \langle q_{i-s}, f \rangle$

$$= \begin{cases} 0 & \text{if } i<n \\ \langle q_{i-n}, f \rangle & \end{cases}$$

$f = q_{i-n}$. $\quad \langle q_n^\perp(q_i), q_{i-n} \rangle = 2$.

i.e. $q_n^\perp(q_i) = 2 q_{i-n}$ if $n>0$

$n=0 \quad q_0 = 1 \quad q_0^\perp = id$

~~Prop. 9.1~~ $\beta_n^{(f)} = \sum_{i \in \mathbb{Z}} (-1)^i q_{n+i}\, q_i^{\perp}(f)$

$\beta_n : \Delta \to \Delta$

define $Q_\lambda = \beta_{\lambda_1} \circ \beta_{\lambda_2} \circ \cdots \circ \beta_{\lambda_\ell}(1)$

Prop 9.1
1) $\beta_s \beta_t + \beta_t \beta_s = 0 \qquad s+t \neq 0$
2) $\beta_s \beta_{-s} + \beta_{-s} \beta_s = (-1)^s \cdot 2\, \mathrm{id}$
3) $\beta_s^2 = 0 \qquad s \neq 0$
4) $\beta_0^2 = 1$

$$\beta_s \beta_t + \beta_t \beta_s = \delta_{s+t,0} (-1)^s \cdot 2\, \mathrm{id}$$

Proof later

α: seq of integers, define

$$y(\alpha) = \begin{cases} 0 & \text{o.w} \\ \text{or, if } \forall n \in \mathbb{N}. \text{ the subseq of } \alpha \text{ with abs value } n \text{ has the form} \end{cases}$$

$-n, n, -n, \ldots, n$

$n, -n, n, -n, \ldots, n$

Then let w be a permutation s.t. $w \cdot \alpha$ has the form

$\lambda, \beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(k)}, 0, \ldots, 0$

λ: Strict partion $(\in \mathcal{D})$. $\beta^{(i)} = (-s_i, s_i)$

then

$$y(\alpha) = \mathrm{sgn}(w) (-1)^{s_1 + \cdots + s_k} \cdot 2^k$$

$\mathrm{str}\,\alpha := \lambda$

③

Th 9.2 $\quad Q_{\alpha} = y(\alpha)\, Q_{\mathrm{str}\,\alpha}$

"Proof": $\quad \beta_{-s}(1) = 0 \qquad s > 0$

$$\sum q_{-s+i}\, q_i^{\perp}(1) = 0$$

$$\beta_{-s}\, \beta_s^{(1)} = (-1)^s \cdot 2\, \mathrm{id}$$

Prop. 9.3 $\quad q_n^{\perp} \beta_r = \beta_r q_n^{\perp} + 2 \sum_{p=1}^{n} \beta_{r-p}\, q_{n-p}^{\perp}$

Prop. 9.4 $\quad q_n^{\perp} Q_{\alpha} = \sum 2^{\eta(\beta)}\, Q_{\alpha-\beta}$

Summation over: $\beta \in \mathbb{N}^{l} \quad l = l(\alpha)$

$\eta(\beta)$: # of entries in β > 0

Proof of 9.3

$$q_n^{\perp} \beta_r(f) = q_n^{\perp} \sum_i (-1)^i\, q_{r+i} \cdot q_i^{\perp}(f)$$

$$q_n^{\perp}(FG) = \sum_i q_{n-i}^{\perp}(F)\, q_i^{\perp}(G)$$

$$= \sum_{m=0}^{n} \sum_i (-1)^i \cdot \underbrace{q_{n-m}^{\perp}(q_{r+i})}_{m=n} \cdot \underbrace{q_m^{\perp}\, q_i^{\perp}(f)}_{0 \le m < n}$$

$$= \sum_i (-1)^i \left(q_{r+i} \cdot q_n^{\perp} q_i^{\perp}(f) + \sum_{m=0}^{n-1} 2\, q_{r+i-n+m} \cdot q_m^{\perp} q_i^{\perp}(f) \right)$$

$$= \sum_i (-1)^i\, q_{r+i}\, q_i^{\perp} q_n^{\perp}(f) + \sum_{p=1}^{n} 2\, q_{r+i-p}\, q_{n-p}^{\perp} q_i^{\perp}(f)$$

$p = n - m$

(cont.) $= \beta_r q_n^{\perp}(f) + 2\sum_{p=1}^{n} \beta_{r-p}\, q_{n-p}^{\perp}(f)$

Proof of 9.4 base case, $\ell(\alpha)=1$. $\alpha=(r)$

$\beta=(n)$ $\eta(\beta)=1.$ $q_n^{\perp}\beta_r(1) = 2\beta_{r-n}$

$q_n^{\perp}\beta_r(1) = \Sigma\, \beta_r q_n^{\perp}(1) + 2\sum_{p=1}^{n} \beta_{r-p}\, q_{n-p}^{\perp}(1)$

$(p=n) = 2\beta_{r-n}$

Proof of Prop. 9.1

$Q(u) = \Sigma\, q_n u^n$

"generating function"

$Q^{\perp}(u) = \Sigma\, q_n^{\perp} u^{-n}$

$\beta(u) = \Sigma\, \beta_n u^n$

Note: $\beta(u) = Q(u)\, Q^{\perp}(-u)$

coefficient of u^n: LHS β_n

RHS: $(\Sigma\, q_i u^i)(\Sigma\, q_j^{\perp}(-1)^j u^{-j})$

$i-j=n$: $\Sigma\, q_i q_j^{\perp}(-1)^j = \Sigma\, q_{j+n}\, q_j^{\perp}(-1)^j = \beta_n$

$i=n+j$

$$\begin{tikzcd}
\Delta(w) \arrow[rr, "\beta"] \arrow[dr, "Q^{\perp}(-u)"'] & & \Delta(u,w) \\
& \Delta(u^-,w) \arrow[ur, "Q(u)"'] &
\end{tikzcd}$$

Also. $\quad Q^{\perp}(v)\, Q(u) = Q(u)\, F(v^{-1}u)$

$$Q(u) F(v^{-1}u)\, Q^{\perp}(v) = Q^{\perp}(v)\, Q(u)\, Q^{\perp}(v) = Q^{\perp}(v)\, Q(u)$$

$$\begin{tikzcd}
\Delta(u^{-}, w) \arrow[rr, "Q(u)"] \arrow[d, "Q^{\perp}(v)"'] & & \Delta(u, w) \arrow[d, "Q^{\perp}(v)"] \\
\Delta(v^{-}, u^{-}, w) \arrow[r, "Q(u)", "\cancel{F(v^{-1}u)}"'] & \Delta(v^{-}, u^{-}, w) \arrow[r, "F(v^{-1}, u)"'] & \Delta(v^{-}, u, w)
\end{tikzcd}$$

let $\phi(u,v) = F(-u^{-1}v) + F(-v^{-1}u) = 1 + \sum_{i=1}^{\infty} (-u^{-1}v)^i + 1 + \sum_{i=1}^{\infty} (-v^{-1}u)^i \; \cancel{\phi(u,v)}$

$$= 2 \sum_i (-u^{-1}v)^i$$

$$\beta(v)\beta(u) = Q(v)\, \underline{Q^{\perp}(-v)\, Q(u)}\, Q^{\perp}(-u)$$

$$= \underline{Q(v)\, Q(u)\, F(-v^{-1}u)}\, Q^{\perp}(-v)\, Q^{\perp}(-u)$$

$$\beta(v)\beta(u) + \beta(u)\beta(v) = \underline{Q(v)\, Q(u)\, \phi(u,v)}\, Q^{\perp}(-v)\, Q^{\perp}(-u)$$

$$= \phi(-v,-u)\, \phi^{\perp}(-v)\, Q^{\perp}(-u)$$

$$= \phi(-v,-u) = \phi(u,v)$$

coeff of $v^i u^j$: $\quad \beta_i \beta_j + \beta_j \beta_i = \begin{cases} 0 & i+j \neq 0 \\ 2(-1)^i \, Id & i+j = 0 \end{cases}$