$T_s^2 = v^{-2} + (1-v^{-2})\, T_s$

eigenvalues $\quad -1, v^{-2} \rightsquigarrow -v, v^{-1}$

$H_s = v T_s$

$H_w = v^{\ell(w)} T_w$

$(H_s + v)(H_s - v^{-1}) = 0$

$\bar{\cdot} : \mathcal{H} \to \mathcal{H}$

$\overline{H_w} = H_{w^{-1}}^{-1}$

$\bar{v} = v^{-1}$

(Th) $\exists \quad \{\underline{H}_w \mid w \in W\}$ a basis of $\mathcal{H}$

$\overline{\underline{H}_w} = \underline{H}_w$

$\underline{H}_w = H_w + \sum_{x<w} h_{x,w} H_x$

$h_{x,w} \in v\mathbb{Z}[v]$ $\quad$ 6

$C_s = \underline{H}_s = H_s + v$

$$H_w . C_s = \begin{cases} H_{ws} + v H_w & ws > w \\ H_{ws} + v^{-1} H_w & ws < w \end{cases}$$

$f \subseteq S$

W_f: parabolic subgp.

W^f: set of minimal coset rep for $W_f \backslash W$ (right)

$\mathcal{L} = \mathbb{Z}[v, v^{-1}]$:

right $\mathcal{H}_f$-mod via $\quad \varphi_u : \mathcal{H}_f \to \mathcal{L}$

$T_s \mapsto u$

is a homomorphism if $u \in \{-v, v^{-1}\}$

$u = v^{-1} \qquad \mathcal{L}(v^{-1})$

$u = -v \qquad \mathcal{L}(-v)$

$M^f = \mathcal{L}(v^{-1}) \otimes_{\mathcal{H}_f} \mathcal{H}$

$N^f = \mathcal{L}(-v) \otimes_{\mathcal{H}_f} \mathcal{H} \qquad$ anti-spherical module

$\mathcal{M}^f$ has a basis $\{M_w := 1 \otimes H_w \mid w \in W^f\}$

$\mathcal{N}^f$ $\{N_w := 1 \otimes H_w \mid w \in W^f\}$

$\overline{\cdot}$ on $\mathcal{M}^f$: $a \in \mathcal{L}$ $x \in \mathcal{H}$

$$\overline{a \otimes x} = \bar{a} \otimes \bar{x}$$

$$\overline{a \otimes yx} = \overline{a \varphi_u(y) \otimes x}$$

$$\bar{a} \otimes \bar{y}\bar{x} = \bar{a}\,\overline{\varphi_u(y)} \otimes \bar{x}$$

$$\varphi_u(\bar{y}) = \overline{\varphi_u(y)}$$

$$C_s = H_s + v$$

$$\varphi_u(C_s) = v^{-1} + v$$

$$\varphi_u(\overline{C_s}) = \overline{\varphi_u(C_s)}$$

$\exists \{\underline{M}_x\}$ basis of $\mathcal{M}^f$

s.t. $\overline{\underline{M}_x} = \underline{M}_x$

and $\underline{M}_x = M_x + \sum_{y<x} m_{y,x} M_y$

$$m_{y,x} \in v\mathbb{Z}[v]$$

$$\underset{(N_x)}{M_x \cdot C_s} = \begin{cases} M_{x.s} + v M_x & \text{if } xs > x \\ M_{x.s} + v^{-1} M_x & \text{if } xs < x \end{cases} \Bigg\} \; xs \in W^f$$
$$\qquad\qquad\qquad \begin{matrix} v^{-1} + v \\ (0) \end{matrix} \quad \text{if } xs \notin W^f$$

$$(M^f)^* = \mathrm{Hom}_{\mathcal{L}}(M^f, \mathcal{L}) \quad \text{left } \mathcal{H}\text{-mod}$$

$$F \in \mathrm{Hom}_{\mathcal{L}}(M^f, \mathcal{L}), \quad x \in \mathcal{H} \quad m \in M^f$$

$$(x.F)(m) = F(m.x)$$

$$\overline{F}(m) = \overline{F(\overline{m})}$$

$$\overline{x.F} = \overline{x}.\overline{F}$$

$$\left(\overline{m.x} = \overline{m}.\overline{x} \right)$$

$$(M_x)^*(M_y) = \delta_{x,y}$$

$$M^x = (M_x)^* (-1)^{\ell(x)}$$

$$\underline{M}^x(\underline{M}_y) = \delta_{x,y}$$

Claim: $\underline{M}^x$ is the KL basis.

$$\underline{M}^x = M^x + \sum_y m^{y,x} M^y$$

$$\sum_x m^{y,x} m_{z,x} (-1)^{\ell(y)+\ell(x)} = \delta_{y,z}$$

- $A \times A \subsetneq A$
- $A \subseteq \hat{A}$

$S_2 \times S_3 \subset S_5$

canonical basis of $V^{\otimes d}$ $\quad V = \mathbb{C}(q)^{\oplus 2}$

for $U_q(\mathfrak{sl}_2)$

$$\underset{v_+}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \quad \underset{v_-}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

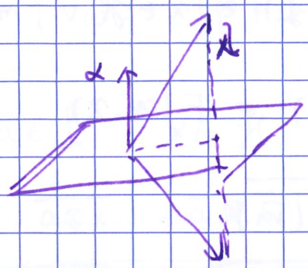
$$v_+ \otimes v_+ \otimes v_+ \otimes v_- \otimes v_-$$

$\widetilde{W} = W \ltimes \mathbb{Z}R$

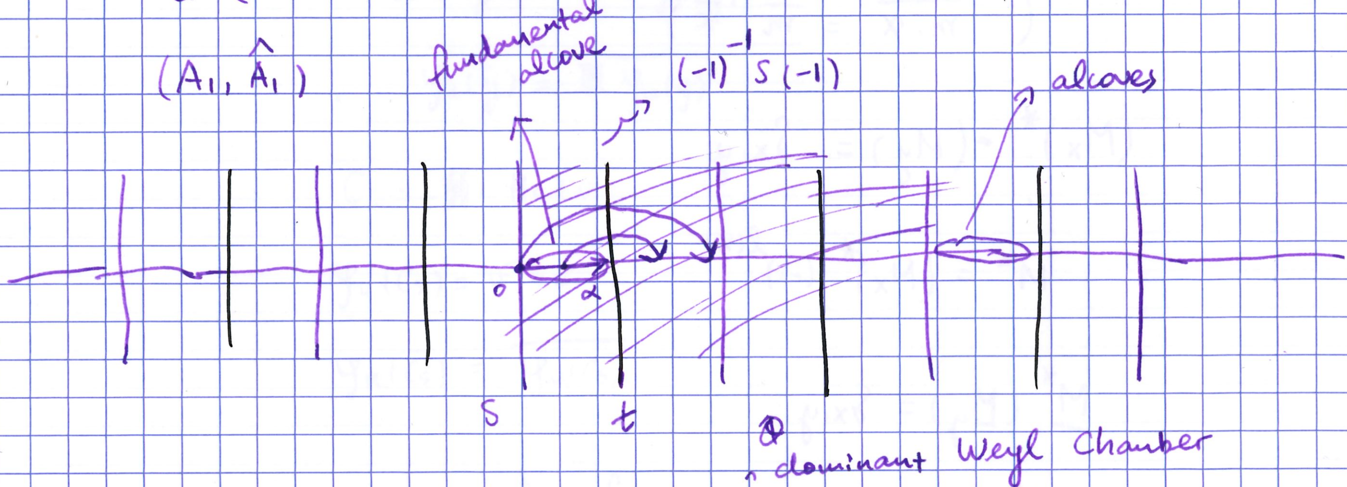
$s_\alpha \cdot \lambda = \lambda - \dfrac{2\langle \alpha^\vee, \lambda \rangle}{\langle \alpha^\vee, \alpha \rangle}\alpha$

$\hat{\widetilde{W}} = W \ltimes X$

extended affine

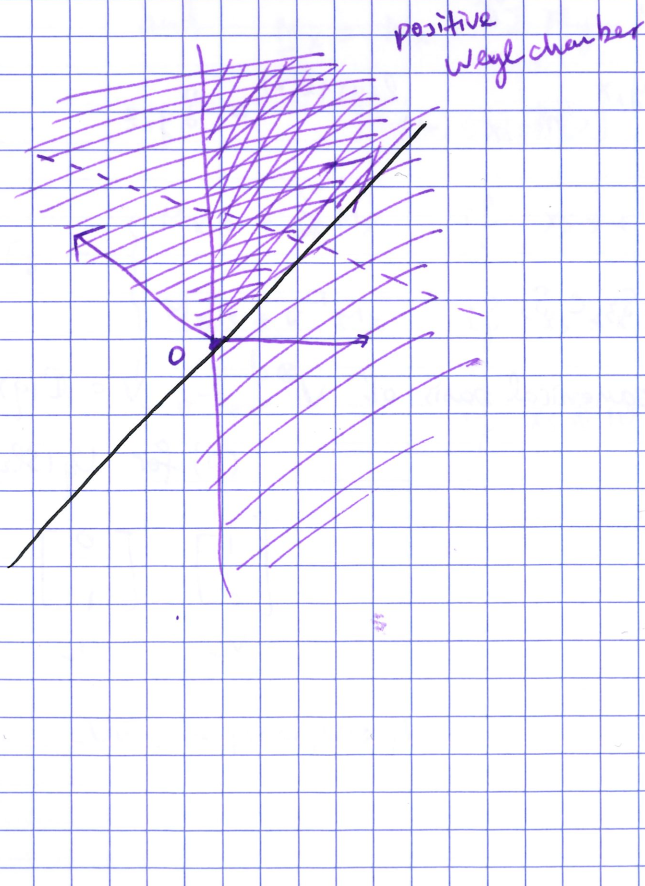


$(A_1, \hat{A}_1)$

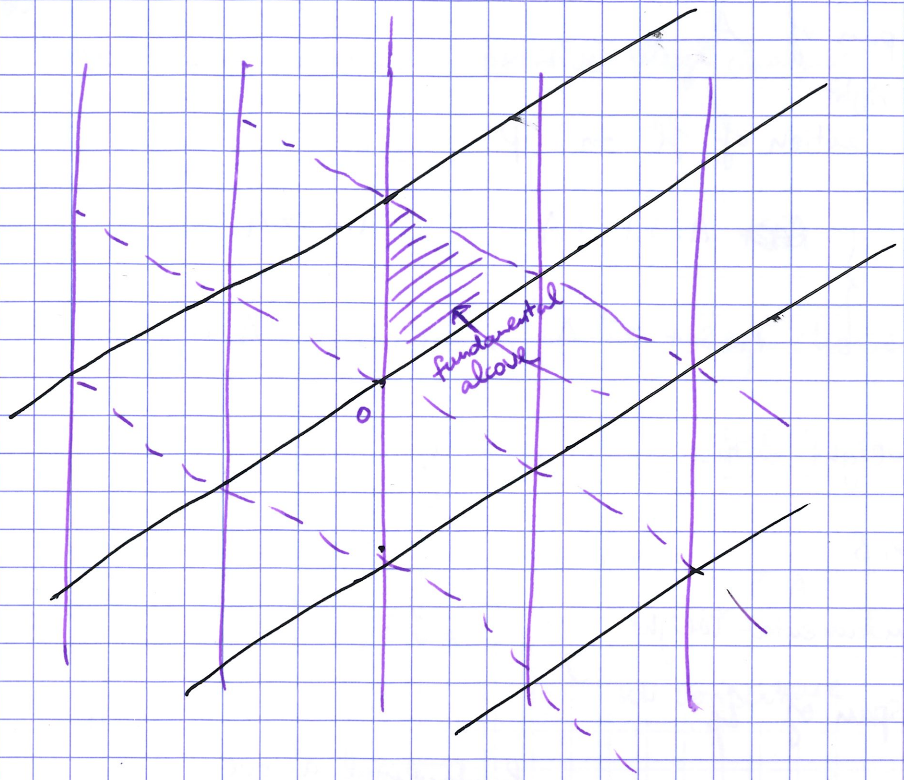


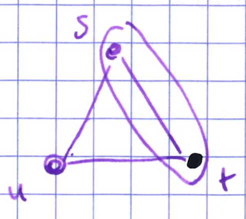
$s \overset{\infty}{—\!\!—} t$

$(A_2, \hat{A}_2)$



$\dfrac{\langle \alpha_1, \alpha_2 \rangle}{\langle \alpha_1, \alpha_1 \rangle} = \dfrac{1}{2}$





F: hyperplane

$$F^c = F^+ \sqcup F^-$$

$\mathcal{A} \curvearrowleft S$

set of alcoves

$A.s$ = the alcove adjacent to A via a hyperplane associated to s

$A.s > A$ $\quad$ $A.s \subset F^+$, $A \subset F^-$

$A.s < A$ $\quad$ ~~A.s~~ otherwise

P: $\mathcal{L}$-span of $\mathcal{A}$

(Th) $\exists$ well-def right action of $\mathcal{H}$ on P

$$A \cdot C_s = \begin{cases} A.s + v A & A.s > A \\ A.s + v^{-1} A & A.s < A \end{cases}$$

X: integral weight lattice

$\langle \alpha^\vee, \varepsilon_\beta \rangle = \delta_{\alpha,\beta}$

ε_β: fundamental weight

X: $\mathbb{Z}$-span of ε_β.

$\lambda \in X$

$$E_\lambda = \sum_{w \in W} (\lambda + w.A^+)(v)^{\ell(w)}$$

A^+: fundamental alcove

P°: $\mathcal{L}$-span of E_λ

(Th) $\exists \; \bar{\cdot} : P^\circ \to P^\circ$

$$\overline{E_\lambda} = E_\lambda$$

$A <_{s_F} B$ if

A, B separated by F.

$<$: transitive closure of $<_{s_F}$ $\forall F$

(Th) $\exists \; \{P_A\}$ basis of P°

$$\overline{P_A} = P_A$$

$$P_A = A + \sum_{B < A} h_{B,A} B$$

$(W, \mathcal{W})$:

$W^f \longleftrightarrow$ alcoves in positive Weyl chamber

$h_{x,y} \rightsquigarrow h_{A,B}$

$x \mapsto A$

simples $\longleftrightarrow$ alcoves in positive Weyl chamber

$$[L_A] = \sum_B \cancel{h_{B,A}}\; m_{B,A} \, [\nabla_B] \, (-1)^{d(A,B)}$$

$$(P_A : \nabla_B) = [\nabla_B : L_A] = m^{B,A}$$

KL conjecture

$$\|$$

$$\boxed{(T_{\hat{A}} : \nabla_B) = n_{B,A}}$$

Soergel's conjecture