§ 5

$G\times H$, $G\tilde{\times}H$, $G\tilde{\times}H$ (σ is zero on G), $G\tilde{Y}H$

goal: irred mod for $G\tilde{Y}H$ (σ nonzero on both)

Recall (G, Z, σ) M: module ($R: G \to GL(M)$)

The associate of M: $R^a(g) = (-1)^{\sigma(g)} R(g)$

- M self-associate $\iff$ $\chi_M = 0$ on $G \setminus \ker\sigma$

Th 4.2 1) M self-associate

$M\downarrow G_0$ has two irreducible constituents which are non-isomorphic, conjugates of each other

2) M: non-self-associate

$M\downarrow G_0$ irreducible ~~con~~ isomorphic to its conjugate

$M\otimes N$: mod for $G\times H$

$(g,h)(u\otimes v) = gu\otimes hv$

Th. All irreducibles for $G\times H$ are of the form $M\otimes N$

M: irreducible for G, N: irreducible for H

$M_1\otimes N_1 \simeq M_2\otimes N_2 \iff M_1\simeq M_2,\ N_1\simeq N_2$

(Use. matrix for $M\otimes N$

= Kronecker product of (matrix for M) $\cdot$ (matrix for N)

Corollary assume $\sigma = 0$ on G $\qquad G\tilde{Y}H = G\tilde{\times}H / \langle (z_1, z_2)\rangle$, $\overset{z}{\shortparallel}$

all irred mod for $G\tilde{\times}H$ arise this way

it is an (irred) mod for $G\tilde{Y}H \iff (z_1, z_2)$ acts as 1

$\iff$ both M, N positive modules / negative mod

$M\otimes N$: negative mod for $G\tilde{Y}H$ $\quad (z_1, 1) \sim z$ acts as -1

$\iff M, N$ both negative $\qquad M_1\otimes N_1 \simeq M_2\otimes N_2 \iff M_1\simeq M_2,\ N_1\simeq N_2$

S: mod for $\ker\sigma$

- $S\uparrow G$ has the matrix

$$S\uparrow G(g) = \begin{bmatrix} \dot{S}(g) & \dot{S}(g_1^{-1}g) \\ \dot{S}(gg_1) & \dot{S}(g_1^{-1}gg_1) \end{bmatrix} \qquad \dot{S}(g) = \begin{cases} S(g) & g \in \ker \\ 0 & \text{o.w.} \end{cases}$$

i.e. $g \in \ker\sigma$ $\quad S\uparrow G(g) = \begin{bmatrix} \dot{S}(g) & 0 \\ 0 & S(g_1^{-1}gg_1) \end{bmatrix}$

$g \notin \ker\sigma$ $\quad S\uparrow G(g) = \begin{bmatrix} 0 & S(g_1^{-1}g) \\ S(gg_1) & 0 \end{bmatrix}$

Check ① is a rep.

$$S\uparrow G(gh) = S\uparrow G(g)\cdot S\uparrow G(h)$$

② has the character

$$\dot{\chi}(g) + \dot{\chi}(g_1^{-1}gg_1)$$

$$\dot{\chi}(g) = \begin{cases} \chi(g) & g \in \ker\sigma \\ 0 & \text{o.w.} \end{cases}$$

if $g_2 = g_1 h$ $\quad h \in \ker\sigma$

$$\dot{\chi}(g_2^{-1}gg_2) = \dot{\chi}(h^{-1}g_1^{-1}gg_1h) = \dot{\chi}(g_1^{-1}gg_1)$$

$L\uparrow G$: is a v.s. $L_{\bar{0}} \oplus L_{\bar{1}}$ s.t.

① $L_{\bar{i}} \cong L$ as mods for $\ker\sigma$

② $g_1 \notin \ker\sigma$

$$g_1 L_{\bar{i}} \subseteq L_{\overline{i+1}}$$

Assume M, N: neg mods

define $M \tilde{\otimes} N$: mod for $G \tilde{\times} H$ (and $G \tilde{Y} H$)

(M,N)
$\boxed{(SA, SA)}$ ←
$\boxed{(SA, NSA)}$
(NSA, SA)
$\boxed{(NSA, NSA)}$ ←

Remark: ① overlap

② well-defined up to associates

① in all cases except for M: NSA N: SA

Let $M \tilde{\otimes} N = (L_{\bar{0}} \oplus L_{\bar{1}}) \otimes N$

$$(g,h)(u \otimes v) = (-1)^{\sigma_H(h)\tau(u)} \; gu \otimes h.v$$

where $\tau(u) = \begin{cases} 0 & \text{if } u \in L_{\bar{0}} \\ 1 & \text{if } u \in L_{\bar{1}} \end{cases}$

Here L: one irreducible constituent of $M \downarrow G_0$

$L \uparrow G = L_{\bar{0}} \oplus L_{\bar{1}}$

② in all cases except for M: SA N: NSA

$M \tilde{\otimes} N = M \otimes (K_{\bar{0}} \oplus K_{\bar{1}})$

$$(g,h)(u \otimes v) = (-1)^{\sigma_G(g)\tau(v) + \sigma_G(g)\sigma_H(h)} \; gu \otimes hv$$

$\tau(v) = \begin{cases} 0 & \text{if } v \in K_{\bar{0}} \\ 1 & \text{if } v \in K_{\bar{1}} \end{cases}$

K: ... $N \downarrow H_0$. $K \uparrow H = K_{\bar{0}} \oplus K_{\bar{1}}$

Goal: compute $\langle \chi_{M \tilde{\otimes} N}, \chi_{M \tilde{\otimes} N} \rangle = 1$. "$\langle \chi_1 + \chi_2, \chi_1 + \chi_2 \rangle = 2$"

Proof: Assume ①. matrix for (g,h)

$$g \in \ker \sigma_G \quad \begin{array}{cc} L_{\bar{0}} \otimes N & L_{\bar{1}} \otimes N \end{array}$$
$$\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$$

$\chi_{M \tilde{\otimes} N} = 0$

$g \in \ker \sigma_G$

kronecker product

$$(g,h) \text{ acts by } \begin{bmatrix} R(g) \otimes S(h) & 0 \\ 0 & (-1)^{\sigma(h)} R(g g_1 g^{-1}) \otimes S(h) \end{bmatrix}$$

sign $(-1)^{\sigma(h)\tau(h)}$

$$\sum_{\substack{g \in \ker \sigma_G \\ h \in H}} |G|^{-1} |H|^{-1} \left| \chi_L(g) \cdot \chi_N(h) + (-1)^{\sigma(h)} \chi_L^c(g) \cdot \chi_N(h) \right|^2$$

$$= \sum_{\substack{g \in \ker \sigma_G \\ h \in \cancel{\ker \sigma_H}\, H}} |G|^{-1} \cancel{|H|^{-1}} \left| \chi_L(g) + (-1)^{\sigma(h)} \chi_L^c(g) \right|^2 |H|^{-1} |\chi_N(h)|^2$$

$$= \underbrace{\sum_{g \in \ker \sigma_G} |G|^{-1} |\chi_L(g) + \chi_L^c(g)|^2}_{\text{"A"}} \cdot \underbrace{\sum_{h \in \ker \sigma_H} |H|^{-1} |\chi_N(h)|^2}_{\text{"B"}}$$

$$+ \underbrace{\sum_{g \in \ker \sigma_G} |G|^{-1} |\chi_L(g) - \chi_L^c(g)|^2}_{\text{"C"}} \cdot \underbrace{\sum_{h \notin \ker \sigma_H} |H|^{-1} |\chi_N(h)|^2}_{\text{"D"}}$$

1 $A = \sum_{g \in \ker \sigma_G} 2^{-1} |\ker \sigma_G|^{-1} |\chi_L(g) + \chi_L^c(g)|^2 = 2^{-1} \langle \chi_L + \chi_L^c, \chi_L + \chi_L^c \rangle$
$= 2^{-1} \cdot 2 = 1$

$C = \quad (+ \to -) \quad = 1$

$D = 0 :$

$B = \sum_{h \in H} |H|^{-1} |\chi_N(h)|^2 = \langle \chi_{N \downarrow H_0}, \chi_{N \downarrow H_0} \rangle\, 1$

② $A = C = 1$

$AB + CD = B + D = 1$

③ M: NSA   N: NSA

$$A = \sum_{g \in \ker\sigma_G} 2^{-1} |\ker\sigma_G|^{-1} \, 2^2 |\chi_L(g)|^2$$

$$= 2 \, \langle \chi_L, \chi_L \rangle = 2$$

$C = 0$

$$B = \sum_{h \in \ker\sigma_H} 2^{-1} |\ker\sigma_H|^{-1} |\chi_N(h)|^2$$

$$= 2^{-1} \langle \chi_N \downarrow H_0, \chi_N \downarrow H_0 \rangle = \frac{1}{2}$$