1 some systems of equations

$$\begin{cases} x+y=3 \\ 2x-y=1 \end{cases} \qquad x+z=0 \qquad \begin{cases} x+z=0 \\ 2x+2z=0 \end{cases}$$

$$(*) \begin{cases} 3x_1-4x_2+x_3=0 \\ x_1+3x_2-x_3=1 \\ x_1-2x_2+x_3=2 \end{cases} \qquad \begin{cases} x=1 \\ x=2 \end{cases}$$

3 The augmented matrix of the systems of eq:

$$\begin{cases} x_1-2x_3+x_2+1=0 \\ x_2-3x_1-1=3 \end{cases} \Rightarrow \begin{cases} x_1+x_2-2x_3=-1 \\ -3x_1+x_2=4 \\ (+0\cdot x_3) \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & -1 \\ -3 & 1 & 0 & 4 \end{array}\right]$$

$(*)$ from 1 : $\left[\begin{array}{ccc|c} 3 & -4 & 1 & 0 \\ 1 & 3 & -1 & 1 \\ 1 & -2 & 1 & 2 \end{array}\right]$

2 ~~3~~ Some Matrices

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 & 1 \\ 0 & \pi & \frac{3}{4} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

④ Identify the elementary row operations performed.:

$$\begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & 2 & 1 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & 4 & 2 & 8 \end{bmatrix}$$ (Scaling)

$$\begin{bmatrix} 3 & 1 & 2 \\ 4 & 3 & 7 \\ 0 & -1 & \frac{3}{4} \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & \frac{3}{4} \\ 4 & 3 & 7 \end{bmatrix}$$ (interchange)

$$\begin{bmatrix} 3 & 1 & 0 & -1 \\ -1 & 2 & 1 & 7 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 1 & 0 & -1 \\ 2 & 3 & 1 & 6 \end{bmatrix}$$ (replacement)

~~⑤ Use elementary row operations to solve eq. set~~

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & -2 \end{bmatrix}$$ scaling

$$\begin{bmatrix} 2 & 0 & 1 & 10 \\ 0 & -1 & 2 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 0 & 1 & 10 \\ 4 & -1 & 4 & 23 \end{bmatrix}$$ replacement

~~8~~ 6 Identify if the following matrices are in row echelon form

$$\begin{bmatrix} 3 & 4 & 3 & 1 & 2 \\ 2 & 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 2 & 4 \end{bmatrix} \quad \text{NOT}$$

pivots

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ & 1 & 0 & 3 \\ & & 1 & 1 \end{bmatrix} \quad \text{Yes}$$

$$\begin{bmatrix} 3 & 1 & -1 & 0 & 4 \\ 0 & 0 & 2 & 1 & 4 \\ 0 & 1 & 0 & 2 & 3 \end{bmatrix} \quad \text{NOT}$$

pivots

$$\begin{bmatrix} 2 & 3 & 2 & 3 \\ & 3 & 0 & 4 \\ & & 1 & 2 \end{bmatrix} \quad \text{YES}$$

7 Perform one elementary matrix operation ~~to~~ on the following matrices, to obtain its row echelon form. and ~~identify~~ the

pivots

$$\begin{bmatrix} 0 & 0 & 3 & 2 & 1 \\ 4 & 7 & 8 & 0 & 9 \end{bmatrix} \xrightarrow{\text{interchange}} \begin{bmatrix} 4 & 7 & 8 & 0 & 9 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix}$$

~~8 Identify the pivots~~

pivots

$$\begin{bmatrix} 1 & 4 & 0 & -1 \\ 2 & -1 & 3 & 8 \end{bmatrix} \xrightarrow{\text{replace}} \begin{bmatrix} 1 & 4 & 0 & -1 \\ 0 & -9 & 3 & 10 \end{bmatrix} \times(-2) \quad \boxed{-2 \quad -8 \quad 0 \quad 2}$$

pivots

$$\begin{bmatrix} 0 & 2 & 9 & 7 \\ 0 & 3 & 1 & 8 \end{bmatrix} \xrightarrow{\text{replace}} \begin{bmatrix} 0 & 2 & 9 & 7 \\ 0 & 0 & -\frac{25}{2} & -\frac{5}{2} \end{bmatrix} \times\left(-\frac{3}{2}\right) \quad [0 \quad -3 \quad -\frac{27}{2} \quad -\frac{21}{2}]$$

(8) identify the pivots in all echelon forms

(9) Use the row reduction algorithm, convert the following matrix to its row echelon form.

9)
$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

Row 1 $\cdot$ (-5) + Row 3 $\rightarrow$ replace Row 3

$[-5 \quad 10 \quad -5 \quad 0] + [5 \quad 0 \quad -5 \quad 10]$ new Row 3

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

Row 2 $\cdot$ (-5): $[0 \quad -10 \quad 40 \quad -40]$

add & replace Row 3:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$ ✓

10) (10)
$$\begin{cases} x_1 - 2x_2 = 1 \\ -x_1 + 3x_2 = 2 \end{cases} \quad \& \quad \begin{cases} x_1 - 2x_2 = 1 \\ x_2 = 3 \end{cases}$$

has the same solution set. $x_1 = 7, x_2 = 3$ or $(7, 3)$

$$\begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ & 1 & 3 \end{bmatrix}$$

(11) Use calculations in # 9. Solve the SLE: (using back substitution)

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ 5x_1 - 5x_3 = 10 \end{cases} \iff \begin{cases} x_1 - 2x_2 + 3x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ 30x_3 = -30 \end{cases}$$

$x_3 = -10$, $\quad 2x_2 + 80 = 8$, $\quad 2x_2 = -72$, $\quad x_2 = -36$

$x_1 + 72 - 30 = 0$, $\quad x_1 = -42$

$(-42, -36, -10)$ is the only solution

(12) Given the following row echelon forms, determine the number of solutions of the associated SLE.

After deleting rows with all zero entries

$$\begin{bmatrix} 3 & 2 & 7 \\ & 1 & -8 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 & 0 & 1 \\ & 2 & 3 & 4 \\ & & 1 & 8 \end{bmatrix}$$

\# Row − # Col = 1

unique sol $\qquad$ unique sol

\# Row − # col > 1

$$\begin{bmatrix} 3 & 2 & 4 & 7 \\ & & 1 & 2 \end{bmatrix}$$

: $x_3 = 2$

$3x_1 + 2x_2 + 4x_3 = 7$

$3x_1 + 2x_2 + 8 = 7$

$3x_1 + 2x_2 = -1$

infinitely many sol.

e.g. $(0, -\frac{1}{2}, 2)$, $(1, -2, 2)$ ...

$$\begin{bmatrix} 4 & 2 & 3 \\ & 3 & -1 \\ & & 1 \end{bmatrix}$$

$0 = 1$ $\quad$ no solutions. $\qquad$ # Row − # Col = 0

~~These are all~~

(13) Vectors with real entries: $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$, $\begin{bmatrix} \pi \\ 2 \\ \frac{1}{4} \\ -3 \end{bmatrix}$, ...

of height 3   of height 4

(14) For $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$ $\quad \vec{w} = \begin{bmatrix} -1 \\ 2 \\ \frac{1}{3} \end{bmatrix}$

Calculate $\vec{v} + \vec{w} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ \frac{13}{3} \end{bmatrix}$

$\frac{1}{2}\vec{w} = \frac{1}{2}\begin{bmatrix} -1 \\ 2 \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{6} \end{bmatrix}$

$2\vec{v} + 3\vec{w}$

$= 2\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 2 \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 8 \end{bmatrix} + \begin{bmatrix} -3 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 9 \end{bmatrix}$

(15) e.g. $3\vec{a} + 2\vec{b}$, $\vec{a} - \vec{c}$, $3\vec{b} + 2\vec{c} + \vec{a}$ are all elements of span $\langle \vec{a}, \vec{b}, \vec{c} \rangle$. In other words, ~~they~~ ~~are~~ each of them is a linear combination of $\vec{a}, \vec{b}, \vec{c}$

(16) If $\vec{u} = \begin{bmatrix} 1 \\ 5 \\ \frac{14}{3} \end{bmatrix}$ ~~and $\vec{u} = x_1\vec{v} + x_2\vec{w}$~~

~~Set up the SLE ~~that~~ with variables x_1, x_2~~

~~$x_1 \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$~~

(16) If $\vec{v} = \begin{bmatrix}-1\\2\\0\end{bmatrix}$ $\vec{w} = \begin{bmatrix}3\\4\\-7\end{bmatrix}$

Set up a SLE with variables x_1 & x_2 such that & identify its augmented matrix

$$\begin{bmatrix}5\\10\\-14\end{bmatrix} = x_1\begin{bmatrix}-1\\2\\0\end{bmatrix} + x_2\begin{bmatrix}3\\4\\-7\end{bmatrix}$$

$$\begin{bmatrix}5\\10\\-14\end{bmatrix} = \begin{bmatrix}-x_1\\2x_1\\0\end{bmatrix} + \begin{bmatrix}3x_2\\4x_2\\-7x_2\end{bmatrix}$$

$$= \begin{bmatrix}-x_1+3x_2\\2x_1+4x_2\\-7x_2\end{bmatrix}$$

$$\begin{cases}-x_1+3x_2=5\\x_1+4x_2=10\\-7x_2=-14\end{cases} \qquad \begin{bmatrix}-1 & 3 & 5\\1 & 4 & 10\\0 & -7 & 14\end{bmatrix}$$

(17) ~~Write~~ Solve for x_1 & x_2 in $\vec{u} = \begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}x_1 + \begin{bmatrix}-1\\\frac{3}{4}\end{bmatrix}x_2$, i.e. write $\vec{u}$ as a linear combination of $\vec{v}$ & $\vec{w}$

augmented matrix : $\begin{bmatrix}2 & -1 & 3\\-1 & \frac{3}{4} & 4\end{bmatrix}$

↳ row echelon form

Row 1 $\times \frac{1}{2}$ ~~+ Row 2~~ : keep

$\begin{bmatrix}1 & -\frac{1}{2} & \frac{3}{2}\end{bmatrix}$ + Row 2 & replace

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & \frac{1}{4} & \frac{11}{2} \end{bmatrix}$$

Solve: $\frac{1}{4}x_2 = \frac{11}{2}$ $\quad x_2 = 22$

$2x_1 - 1 \cdot 22 = 3$

$x_1 = \frac{25}{2}$.

$\vec{u} = \frac{25}{2}\vec{v} + 22\vec{w}$

* updated 2/5

Solutions to examples

(18)

$$\begin{bmatrix} 3 & 4 & 1 & 0 \\ 1 & 2 & -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 4 & 1 & 0 \end{bmatrix} \Rightarrow \begin{matrix} [-3 & -6 & 3 & 0] \\ \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -2 & 4 & 0 \end{bmatrix} \end{matrix}$$

$-2x_2 + 4x_3 = 0$

$-2x_2 = -4x_3$

$x_2 = 2x_3$

$x_1 + 2(2x_2) - x_3 = 0.$

$x_1 = -x_3$

$x_1 = -x_3,\ x_2 = 2x_3,\ x_3 = x_3$

is the solution

$$\vec{x} = \begin{bmatrix} -x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

The solutions (the solution set) are all linear combinations of the vector $\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$

i.e. the solution set is $\text{Span} \left\langle \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \right\rangle$

(19)

$$\begin{bmatrix} -1 & 2 & 3 & 4 & 0 \\ 5 & 1 & 0 & -1 & 0 \end{bmatrix} \Rightarrow \begin{matrix} [-5 & 10 & 15 & 20 & 0] \\ \begin{bmatrix} -1 & 2 & 3 & 4 & 0 \\ 0 & 11 & 15 & 19 & 0 \end{bmatrix} \end{matrix}$$

$11x_2 + 15x_3 + 19x_4 = 0$

$x_2 = -\frac{15}{11}x_3 - \frac{19}{11}x_4$

$-x_1 + 2\left(-\frac{15}{11}x_3 - \frac{19}{11}x_4\right) + 3x_3 + 4x_4 = 0$

$-x_1 + \frac{3}{11}x_3 + \frac{6}{11}x_4 = 0$

$x_1 = \frac{3}{11}x_3 + \frac{6}{11}x_4$

$$\vec{x} = \begin{bmatrix} \frac{3}{11}x_3 + \frac{6}{11}x_4 \\ -\frac{15}{11}x_3 - \frac{19}{11}x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{3}{11}x_3 \\ -\frac{15}{11}x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{6}{11}x_4 \\ -\frac{19}{11}x_4 \\ 0 \\ x_4 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} \frac{3}{11} \\ -\frac{15}{11} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \frac{6}{11} \\ -\frac{19}{11} \\ 0 \\ 1 \end{bmatrix}$$

The solution set is all linear combinations of the vectors

$$\begin{bmatrix} \frac{3}{11} \\ -\frac{15}{11} \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} \frac{6}{11} \\ -\frac{19}{11} \\ 0 \\ 1 \end{bmatrix}$$

or, the solution set is $\text{span}\left\langle \begin{bmatrix} \frac{3}{11} \\ -\frac{15}{11} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{6}{11} \\ -\frac{19}{11} \\ 0 \\ 1 \end{bmatrix} \right\rangle$

Alternatively,

$$x_3 = -\frac{15}{11}x_2 - \frac{19}{11}x_4$$

$$x_1 = 2(\;\; 2x_2 + 3\left(-\frac{15}{11}x_2 - \frac{19}{11}\right)x_4 + 4x_4$$

$$= \;\; -\frac{23}{11}x_2 - \frac{16}{11}x_4$$

$$\text{span}\left\langle \begin{bmatrix} -\frac{23}{11} \\ 1 \\ -\frac{15}{11} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{16}{11} \\ 0 \\ -\frac{19}{14} \\ 1 \end{bmatrix} \right\rangle$$

(19) Revisted

$$11x_2 + 15x_3 + 19x_4 = 0$$

guess. : $x_2 = 0$. $x_3 = 1$ $x_4 = -\cancel{\frac{19}{15}}\ -\frac{15}{19}$

$x_2 = 1$ $x_3 = 0$ $x_4 = -\frac{11}{19}$

$$\vec{x} = \begin{bmatrix} -\frac{9}{19} \\ 0 \\ 1 \\ -\frac{15}{19} \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} -\frac{6}{19} \\ 1 \\ 0 \\ -\frac{11}{19} \end{bmatrix}$$

$$x_1 = 2x_2 + 3x_3 + 4x_4$$

the Solution set is span $\left\langle \begin{bmatrix} -\frac{9}{19} \\ 0 \\ 1 \\ -\frac{15}{19} \end{bmatrix}, \begin{bmatrix} -\frac{6}{19} \\ 1 \\ 0 \\ -\frac{11}{19} \end{bmatrix} \right\rangle$

(20) 2) $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0} = x_1\vec{v_1} + x_2\vec{v_2} + x_3\vec{v_3}$

has augmented matrix

$$\begin{bmatrix} 3 & 7 & 1 & 0 \\ -1 & -4 & -2 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & -4 & -2 & 0 \\ 3 & 7 & 1 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix}$$

$[-3 \ \ -12 \ \ -6 \ \ 0]$ $\qquad$ $[0 \ \ -3 \ \ -3 \ \ 0]$

$$\Rightarrow \begin{bmatrix} -1 & -4 & -2 & 0 \\ 0 & -5 & -5 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & -4 & -2 & 0 \\ 0 & -5 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3 variables & 2 equations

infinitely many solutions

the vectors are NOT linearly independent

/ linearly DEPENDENT

(* updated 2/7)

Solutions

(20) 3) $\vec{0} = x_1\vec{v_1} + x_2\vec{v_2} + x_3\vec{v_3}$

$$\begin{bmatrix} 0 & 3 & -1 & 0 \\ 2 & 1 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 2 & 0 \\ 0 & 3 & -1 & 0 \end{bmatrix} \qquad 3x_2 - x_3 = 0$$

3 variables & 2 equations

infinitely many solutions

$\vec{v_1}, \vec{v_2}, \vec{v_3}$ are linearly DEPENDENT

(i.e. NOT linearly independent)

(21) 4) $\vec{0} = x_1\vec{e_1} + x_2\vec{e_2} + x_3\vec{e_3} + x_4\vec{e_4}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

unique solution

$x_1 = 0, \; x_2 = 0, \; x_3 = 0, \; x_4 = 0$

$\vec{e_1} - \vec{e_4}$ are linearly INDEPENDENT

(23)

$$\begin{cases} x_1 = 0.2x_1 + 0.1x_2 + 0.2x_3 \\ x_2 = 0.4x_1 + 0.3x_2 + 0.5x_3 \\ x_3 = 0.4x_1 + 0.6x_2 + 0.3x_3 \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

24 1)

$$f\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + f\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= \begin{bmatrix}3\\4\\2\end{bmatrix} + \begin{bmatrix}-1\\1\\3\end{bmatrix} = \begin{bmatrix}2\\5\\5\end{bmatrix}$$

f is NOT linear

25 2)

$$\begin{bmatrix}-3\\-10\end{bmatrix} = x_1\begin{bmatrix}2\\3\end{bmatrix} + x_2\begin{bmatrix}1\\-4\end{bmatrix}$$

$$\begin{bmatrix}2 & 1 & -3\\3 & -4 & -10\end{bmatrix} \rightarrow \begin{bmatrix}2 & 1 & -3\\0 & -\frac{11}{2} & -\frac{11}{2}\end{bmatrix}$$

$x_2 = 1,$ $\quad 2x_1 + x_2 = -3$ $\quad x_1 = -2$

$$f\left(\begin{bmatrix}-3\\-10\end{bmatrix}\right) = f\left(-2\begin{bmatrix}2\\3\end{bmatrix} + \begin{bmatrix}1\\-4\end{bmatrix}\right)$$

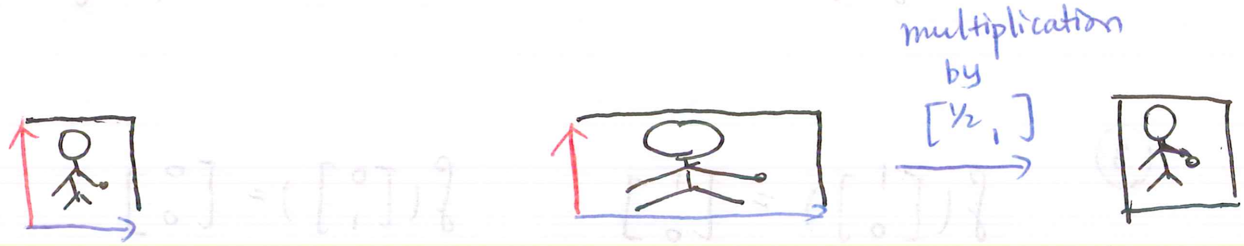
$$= -2f\left(\begin{bmatrix}2\\3\end{bmatrix}\right) + f\left(\begin{bmatrix}1\\-4\end{bmatrix}\right)$$

$$= -2\cdot\begin{bmatrix}7\\0\\-1\end{bmatrix} + \begin{bmatrix}0\\-1\\2\end{bmatrix} = \begin{bmatrix}-14\\-1\\4\end{bmatrix}$$

26

$$f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\cdot 1+0\cdot 0\\0\cdot 1+1\cdot 0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$$

$$f\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2\cdot 0+0\cdot 1\\0\cdot 0+1\cdot 1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$



27

1)

$$f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\cdot\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\cdot 1+0\cdot 0\\0\cdot 1+(-1)\cdot 0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$f\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\cdot\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\cdot 0+0\cdot 1\\0\cdot 0+(-1)\cdot 1\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix}$$

2)

$$f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\cdot\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\cdot 1+2\cdot 0\\0\cdot 1+1\cdot 0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$f\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\cdot\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\cdot 0+2\cdot 1\\0\cdot 0+1\cdot 1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

(2)

$$f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\cdot1+2\cdot0\\0\cdot1+1\cdot0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$f\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\cdot0+2\cdot1\\0\cdot0+1\cdot1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

(3)

$$f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix} \qquad f\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}$$

(28)

1) :

$$\begin{bmatrix}3 & -1\\2 & 4\end{bmatrix}$$

2) :

$$\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix} + \begin{bmatrix}-1\\2\end{bmatrix}$$

$$f\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = f\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) + f\left(\begin{bmatrix}-1\\2\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

$$f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = f\left(\begin{bmatrix}1\\-1\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right) = f\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) + f\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= \begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

matrix for f : $\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}$

* updated 2/18

Solutions

33

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 6 & 14 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 6 & 14 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \cdot (-3) + R_2 \to R_2} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 1 & & \\ -3 & 1 & \\ & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 4 \\ 3 & 6 & 14 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_3 \cdot \frac{1}{2}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & -2 & \\ & 1 & \\ & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & -4 \\ & 1 & \\ & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & & -4 \\ & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & \\ & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & \\ & & \tfrac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & & \\ -3 & 1 & \\ & & 1 \end{bmatrix}}_{\text{"}A^{-1}\text{"}} \cdot \overbrace{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 6 & 14 \\ 0 & 1 & 0 \end{bmatrix}}^{\text{"}A\text{"}} = I$$

$$A^{-1} = \begin{bmatrix} 1 & & -4 \\ & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & \\ & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & \\ & & \tfrac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & & \\ -3 & 1 & \\ & & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & & \\ -3 & 1 & \\ & & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 1 & & \\ & 1 & \\ & & \tfrac{1}{2} \end{bmatrix}^{-1}\begin{bmatrix} 1 & -2 & \\ & 1 & \\ & & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & & -4 \\ & 1 & \\ & & 1 \end{bmatrix}^{-1}$$

Solutions

34

$$\begin{bmatrix} 1 & -2 & -3 \\ 7 & 0 & 7 \\ -2 & -1 & -4 \end{bmatrix} \xrightarrow{R_1 \cdot (-7) + R_2 \curvearrowright R_2} \begin{bmatrix} 1 & -2 & -3 \\ 0 & 14 & 28 \\ -2 & -1 & -4 \end{bmatrix}$$

$$\xrightarrow{R_1 \cdot 2 + R_3 \curvearrowright R_3} \begin{bmatrix} 1 & -2 & -3 \\ 0 & 14 & 28 \\ 0 & -5 & -10 \end{bmatrix} \xrightarrow[R_3/(-5)]{R_2/14} \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_2 \cdot (-1) + R_3 \curvearrowright R_3} \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

A has 2 pivots $\leq$ # rows in A

A is NOT Invertible

35

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 7 & 7 & 0 \\ -1 & -2 & 0 & 0 \end{array}\right] \xrightarrow{\text{Row reduction}}$$

pivot positions:

$$\left[\begin{array}{ccc|c} * & & & \\ & * & & \\ & & * & \end{array}\right]$$

~~linearly independent~~

~~Invertible~~

linearly independent

$$\left[\begin{array}{ccc|c} * & 0 & * & \end{array}\right] \quad \left[\begin{array}{cc|c} * & & \\ & * & \end{array}\right]$$

$$\left[\begin{array}{ccc|c} * & * & & \\ & * & & \\ & & 0 & \end{array}\right] \quad \left[\begin{array}{c|c} * & \end{array}\right]$$

~~NOT linearly independent~~

~~NOT invertible~~

NOT linearly independent

(36)

- $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\begin{cases} 0 - x_2 + 2x_3 = 0 \\ 5x_1 - 6x_2 + x_3 = 0 \\ 5x_1 - 9x_2 + 7x_3 = 0 \end{cases}$$

augmented matrix $\begin{bmatrix} 0 & -1 & 2 & 0 \\ 5 & -6 & 1 & 0 \\ 5 & -9 & 7 & 0 \end{bmatrix}$

~~linear independent~~ has a unique solutions $\Leftrightarrow$ A is invertible i.e. A has three pivots

(37)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{R_1 \cdot (-\frac{c}{a}) + R_2} \begin{bmatrix} a & b \\ 0 & -\frac{bc}{a} + d \end{bmatrix} \xrightarrow{R_1/a} \begin{bmatrix} 1 & b/a \\ 0 & \frac{ad-bc}{a} \end{bmatrix}$$

$$\xrightarrow{R_2 \cdot \left(\frac{a}{ad-bc}\right)} \begin{bmatrix} 1 & b/a \\ & 1 \end{bmatrix} \xrightarrow{R_2 \cdot (-b/a) + R_1 \rightsquigarrow R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_2 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \xrightarrow{R_1 \cdot (-\frac{c}{a}) + R_2} \begin{bmatrix} 1 & 0 \\ -\frac{c}{a} & 1 \end{bmatrix} \xrightarrow{R_1/a} \begin{bmatrix} 1/a & 0 \\ -\frac{c}{a} & 1 \end{bmatrix}$$

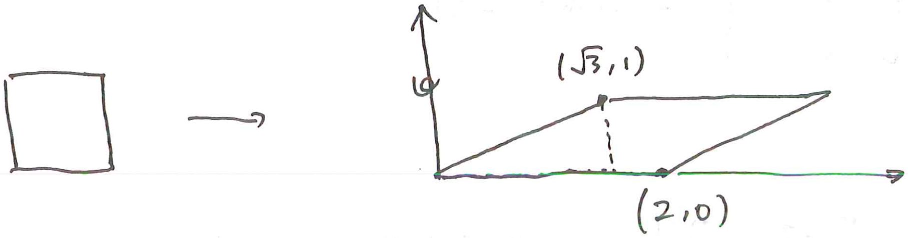
$$\xrightarrow{R_2 \cdot \left(\frac{a}{ad-bc}\right)} \begin{bmatrix} 1/a & 0 \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \xrightarrow{R_2 \cdot (-b/a) + R_1} \begin{bmatrix} \frac{ad}{a(ad-bc)} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(* updated 3/4)

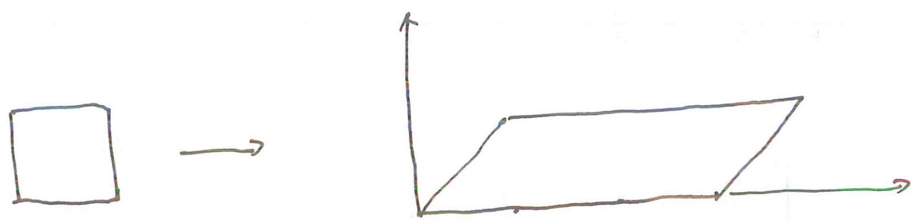
Solutions

(40)



area of ▱ = base · height.

$2 \cdot 1 = 2$ $\quad$ $\det\left(\begin{bmatrix} \sqrt{3} & 2 \\ 1 & 0 \end{bmatrix}\right) = -2$



area $= 3 \cdot 1 = 3$

$\det\left(\begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}\right) = 3$

(41)

$\det A = 4 - 14 = -10$

$\det B = -8 - 0 = -8$

$AB = \begin{bmatrix} 1 & 7 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -3 & 8 \end{bmatrix} = \begin{bmatrix} -22 & 56 \\ -14 & 32 \end{bmatrix}$

$\det(AB) = 80$

$BA = \begin{bmatrix} -1 & 0 \\ -3 & 8 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -1 & -7 \\ 13 & 11 \end{bmatrix}$

$\det(BA) = 80$

(42) Original: $\det \begin{bmatrix} -2 & 7 \\ -4 & 1 \end{bmatrix} = -2 + 28 = 26$

After Scaling: $\det \begin{bmatrix} -2 & 7 \\ -4\pi & \pi \end{bmatrix} = -2\cdot\pi + 28\pi = 26\pi$

After Swapping: $\det \begin{bmatrix} -4 & 1 \\ -2 & 7 \end{bmatrix} = -28 + 2 = -26$

After replacement: $\det \begin{bmatrix} -2 & 7 \\ -2\cdot 314 - 4 & 7\cdot 314 + 1 \end{bmatrix}$

$= -2\cdot 7\cdot 314 - 2 - (-2\cdot 7\cdot 314 - 4\cdot 7)$

$= -2 + 28 = 26$

(43) $\det A = -4 \det \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix} - 7 \det \begin{bmatrix} 3 & -3 \\ 0 & 2 \end{bmatrix} + 11 \det \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

$= -4\cdot(2+6) - 7\cdot 6 + 11\cdot 6$

$= -32 - 42 + 66 = -8.$

A: invertible

(44)

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} \det A_{11} & -\det A_{21} & \det A_{31} \\ -\det A_{12} & \det A_{22} & -\det A_{32} \\ \det A_{13} & -\det A_{23} & \det A_{33} \end{bmatrix}$$

$$= \frac{1}{-8}\begin{bmatrix} \det\begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix} & -\det\begin{bmatrix} 7 & 11 \\ 2 & 2 \end{bmatrix} & \det\begin{bmatrix} 7 & 11 \\ 1 & -3 \end{bmatrix} \\ -\det\begin{bmatrix} 3 & -3 \\ 0 & 2 \end{bmatrix} & +\det\begin{bmatrix} -4 & 11 \\ 0 & 2 \end{bmatrix} & -\det\begin{bmatrix} -4 & 11 \\ 3 & -3 \end{bmatrix} \\ \det\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} & -\det\begin{bmatrix} -4 & 7 \\ 0 & 2 \end{bmatrix} & \det\begin{bmatrix} -4 & 7 \\ 3 & 1 \end{bmatrix} \end{bmatrix}$$

$$= \frac{1}{-8}\begin{bmatrix} 8 & 8 & -32 \\ -6 & -8 & +21 \\ 6 & 8 & \cancel{-32}\ -25 \end{bmatrix}$$

(45)

$$A^T = \begin{bmatrix} -4 & 3 & 0 \\ 7 & 1 & 2 \\ 11 & -3 & 2 \end{bmatrix}$$

$$\det A^T = -4\cdot\det\begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix} - 3\det\begin{bmatrix} 7 & 2 \\ 11 & 2 \end{bmatrix}$$

$$-0$$

$$= -4\cdot 8 - 3\cdot(-8)$$

$$= -32+24 = -8$$

$\cancel{\det A^{-1}} =$

$$A^{-1} = \begin{bmatrix} -1 & -1 & 4 \\ \frac{3}{4} & 1 & -\frac{21}{8} \\ -\frac{3}{4} & -1 & \frac{25}{8} \end{bmatrix}$$

$$\det A^{-1} = -1 \det \begin{bmatrix} 1 & -\frac{21}{8} \\ -1 & \frac{25}{8} \end{bmatrix} \div (-1) \det \begin{bmatrix} \frac{3}{4} & -\frac{21}{8} \\ -\frac{3}{4} & \frac{25}{8} \end{bmatrix}$$

$$+ 4 \det \begin{bmatrix} \frac{3}{4} & 1 \\ -\frac{3}{4} & -1 \end{bmatrix}$$

$$= -1 \cdot \frac{4}{8} + \frac{75-63}{32} + 4 \cdot \cancel{\frac{3}{2}}\ 0$$

$$= -\frac{1}{2} + \frac{12}{32} = \frac{-1}{8}$$

(46) $$\vec{d} = 7\vec{a} - \vec{b} + 4 \cdot (3\vec{a} + 2\vec{b})$$

$$= 7\vec{a} - \vec{b} + 12\vec{a} + 8\vec{b} = 19\vec{a} + 7\vec{b}$$

$$2\vec{b} = \vec{c} - 3\vec{a}, \quad \vec{b} = \frac{1}{2}\vec{c} - \frac{3}{2}\vec{a}$$

$$\vec{d} = 7\vec{a} - \left(\frac{1}{2}\vec{c} - \frac{3}{2}\vec{a}\right) + \cancel{2b}\ 4\vec{c}$$

$$= \frac{11}{2}\vec{a} + \frac{7}{2}\vec{c}$$

(47) Pick $\vec{v}_1$

$\vec{v}_1, \vec{v}_2$ linearly independent.

Test if $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent:

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = 0$$

$$\begin{bmatrix} 1 & 3 & -5 \\ -1 & -1 & -3 \\ 2 & -1 & 18 \end{bmatrix} \xrightarrow[R_3 = R_3 + R_1(-2)]{R_2 = R_1 + R_2} \begin{bmatrix} 1 & 3 & -5 \\ 0 & 2 & -8 \\ 0 & -7 & 28 \end{bmatrix} \xrightarrow{R_3 = R_2 \cdot \frac{7}{2} + R_3} \begin{bmatrix} 1 & 3 & -5 \\ 0 & 2 & -8 \\ 0 & 0 & 0 \end{bmatrix}$$

NOT linearly indep.

(47) cont.)

Test if $\vec{v_1}, \vec{v_2}, \vec{v_4}$ are linearly indep.

$$x_1\vec{v_1} + x_2\vec{v_2} + x_4\vec{v_4} = 0$$

$$\begin{bmatrix} 1 & 3 & 7 \\ -1 & -1 & -1 \\ 2 & -1 & -7 \end{bmatrix} \xrightarrow[R_3 = R_3 + R_1(-2)]{R_2 = R_1 + R_2} \begin{bmatrix} 1 & 3 & 7 \\ 0 & 2 & 6 \\ 0 & -7 & -21 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_2 \cdot \frac{7}{2} + R_3} \begin{bmatrix} 1 & 3 & 7 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

NOT linearly independent.

Alternatively

$$\begin{bmatrix} 1 & 3 & -5 & 7 & 3 \\ -1 & -1 & -3 & -1 & -3 \\ 2 & -1 & 18 & -7 & 6 \end{bmatrix} \xrightarrow[R_2 = R_3 + R_1(-2)]{R_2 = R_1 + R_2} \begin{bmatrix} 1 & 3 & 5 & 7 & 3 \\ 0 & 2 & -8 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2 = # of pivots.

$\vec{v_1}, \vec{v_2}$: maximally linearly independent vectors

* updated 3/11

Solutions.

(51)

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \xrightarrow{R_1 \cdot 3} \begin{bmatrix} 3R_1 \\ R_2 \\ R_3 \end{bmatrix} \xrightarrow{R_2 = R_2 + 4R_1} \begin{bmatrix} 3R_1 \\ R_2 + 12R_1 \\ R_3 \end{bmatrix} \xrightarrow{R_3 = 3R_2 + R_3} \begin{bmatrix} 3R_1 \\ R_2 + 12R_1 \\ R_3 + 3R_2 + 36R_1 \end{bmatrix} = \begin{bmatrix} \\ \\ 0 \; 0 \; 0 \end{bmatrix}$$

$$R_3 + 3R_2 + 36R_1 = 0$$

$$R_3 = -3R_2 - 36R_1$$

(52)

$$\begin{bmatrix} 1 & 7 & 2 & 0 \\ 2 & -1 & 3 & -2 \\ -4 & 0 & 1 & 9 \end{bmatrix} \xrightarrow[R_1 \cdot 4 + R_3 \to R_3]{R_1 \cdot (-2) + R_2 \to R_2} \begin{bmatrix} 1 & 7 & 2 & 0 \\ 0 & -15 & -1 & -2 \\ 0 & 28 & 9 & 9 \end{bmatrix}$$

$$\xrightarrow{R_2 \cdot \left(\frac{28}{15}\right) + R_3 \to R_3} \begin{bmatrix} 1 & 7 & 2 & 0 \\ 0 & -15 & -1 & -2 \\ 0 & 0 & \frac{135-28}{15} & \frac{135-56}{15} \end{bmatrix}$$

$$\frac{135-28}{15} = \frac{107}{15} \qquad \frac{135-56}{15} = \frac{79}{15}$$

$$\frac{107}{15} x_3 + \frac{77}{15} x_4 = 0 \qquad \cancel{\frac{107}{15}} \quad x_3 = -\frac{79}{107} x_4$$

$$-15x_2 - \cancel{x_3} \left(-\frac{79}{107} x_4\right) - 2x_4 = 0$$

$$-15x_2 = \frac{214-79}{107} x_4$$

$$-15x_2 = \frac{135}{107} x_4$$

$$x_2 = -\frac{7}{107} x_4$$

$$x_1 - \frac{7}{107} \cdot 7x_4 + 2 \cdot \left(-\frac{79}{107} x_4\right) = 0$$

$$x_1 = \frac{158+49}{107} x_4 = \frac{207}{107} x_4$$

52) cont.

Solution space / Null space:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{207}{107}x_4 \\ -\frac{7}{107}x_4 \\ -\frac{79}{107}x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} \frac{207}{107} \\ -\frac{7}{107} \\ -\frac{79}{107} \\ 1 \end{bmatrix}$$

$$\text{span}\left\langle \begin{bmatrix} \frac{207}{107} \\ -\frac{7}{107} \\ -\frac{79}{107} \\ 1 \end{bmatrix} \right\rangle$$

\# of pivots = 3

Rank A = 3

~~54~~ 53) Row A is spanned by $\begin{bmatrix} 1 \\ 7 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \\ -2 \end{bmatrix} \begin{bmatrix} -4 \\ 0 \\ 1 \\ 9 \end{bmatrix}$

~~54~~ 53)

$$\det \begin{bmatrix} \frac{207}{107} & 1 & 2 & -4 \\ -\frac{7}{107} & 7 & -1 & 0 \\ -\frac{79}{107} & 2 & 3 & 1 \\ 1 & 0 & -2 & 9 \end{bmatrix} \neq 0$$

linearly independent.

54 Instead, use $\vec{v} = \begin{bmatrix} 2 \\ 7 \\ -1 \\ 3 \end{bmatrix}$

$$\vec{w_1} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \quad \vec{w_2} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 7 \end{bmatrix} \quad \vec{w_3} = \begin{bmatrix} 0 \\ 0 \\ 4 \\ -2 \end{bmatrix} \quad \vec{w_4} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}$$

i.e. $A = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ -2 & 0 & 4 & 2 \\ 1 & 7 & -2 & 1 \end{bmatrix}$

find solution to $A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ -1 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = A^{-1} \begin{bmatrix} 2 \\ 7 \\ -1 \\ 3 \end{bmatrix}$$

A^{-1}: $\det A = 1 \cdot \left(1\,(4+4) - 1 \cdot (0-28)\right) - 3\left(-1 \cdot (4-4)\right)$
$- 1 \cdot \left(-1 \cdot (4-4)\right) = 1 \cdot (8+28) = 36$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} \begin{matrix} 1 \cdot (4+4) \\ -1\,(0-28) \end{matrix} & -1 \cdot (4-4) & \begin{matrix} -1 \cdot (-2-2) \\ -1 \cdot (-14-0) \end{matrix} & -1 \cdot (4-4) \\ \begin{matrix} 3 \cdot (4-(-4)) \\ +1 \cdot (0-28) \end{matrix} & \begin{matrix} 1 \cdot (4+4) \\ +1 \cdot (4-4) \end{matrix} & \begin{matrix} 1 \cdot (0-14) \\ -3 \cdot (-2-2) \\ +1 \cdot (-14-0) \end{matrix} & \begin{matrix} 1 \cdot (0-14) \\ -3 \cdot (4-4) \end{matrix} \\ \begin{matrix} 3 \cdot (0-2) \\ +1 \cdot (-2-0) \end{matrix} & \begin{matrix} 1 \cdot (0-2) \\ +1 \cdot 0 \end{matrix} & \begin{matrix} 1 \cdot (1+7) \\ -3 \cdot (0+1) \\ +1 \cdot (0-1) \end{matrix} & \begin{matrix} 1 \cdot (-2-0) \\ -3 \cdot (0) \end{matrix} \\ \begin{matrix} 3 \cdot (0+4) \\ +1 \cdot (4-0) \end{matrix} & \begin{matrix} 1 \cdot (0+4) \\ +1 \cdot (0-0) \end{matrix} & \begin{matrix} 1 \cdot (2-0) \\ -3\,(0-2) \\ +1 \cdot (0+2) \end{matrix} & \begin{matrix} 1 \cdot (4-0) \\ -3 \cdot (0-0) \end{matrix} \end{bmatrix}^{T}$$

or use Python!

54. (cont).

$$A^{-1} = \frac{1}{36}\begin{bmatrix} 36 & 0 & 18 & 0 \\ -4 & 8 & -16 & -14 \\ -8 & -2 & 4 & -2 \\ 16 & 4 & 10 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{9} & \frac{2}{9} & -\frac{4}{9} & -\frac{7}{18} \\ -\frac{2}{9} & -\frac{1}{18} & \frac{1}{9} & -\frac{2}{18} \\ \frac{4}{9} & \frac{1}{9} & \frac{5}{18} & \frac{1}{9} \end{bmatrix}$$

$$\vec{x} = A^{-1} \cdot \begin{bmatrix} 2 \\ 7 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 - \frac{1}{2} \\ -\frac{2}{9} + \frac{14}{9} + \frac{4}{9} - \frac{21}{18} \\ -\frac{4}{9} - \frac{7}{18} - \frac{1}{9} - \frac{6}{18} \\ \frac{8}{9} + \frac{7}{9} - \frac{5}{18} + \frac{3}{9} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{11}{18} \\ -\frac{23}{18} \\ \frac{31}{18} \end{bmatrix}$$

therefore $\vec{v} = \frac{3}{2}\vec{w_1} + \frac{11}{18}\vec{w_2} - \frac{23}{18}\vec{w_3} + \frac{31}{18}\vec{w_4}$

55)

$$\vec{v}_{\{\vec{e_1}, \vec{e_2}, \vec{e_3}, \vec{e_4}\}} = \begin{bmatrix} 2 \\ 7 \\ -1 \\ 3 \end{bmatrix}$$

$$\vec{v}_{\{\vec{w_1}, \vec{w_2}, \vec{w_3}, \vec{w_4}\}} = \begin{bmatrix} \frac{3}{2} \\ \frac{11}{18} \\ -\frac{23}{18} \\ \frac{31}{18} \end{bmatrix}$$

Solutions

56

1) $\vec{w}_{\{\vec{v}_1, \vec{v}_2\}} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

2) $\vec{w}_{\{\vec{w}_1, \vec{w}_2\}} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$

57

1) No, not enough vectors to span

2) No, not linearly independent

3) No, - - - - - - -

4) No, - - - - - - - - -

5) Yes

58

$$\vec{e}_1\,_{\{\vec{v}_1, \vec{v}_2\}} = \begin{bmatrix} 2 & -1 \\ 7 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 3 & 1 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} 3 \\ -7 \end{bmatrix} = \begin{bmatrix} \frac{3}{13} \\ -\frac{7}{13} \end{bmatrix}$$

$$\vec{e}_2\,_{\{\vec{v}_1, \vec{v}_2\}} = \begin{bmatrix} 2 & -1 \\ 7 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 3 & 1 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{13} \\ \frac{2}{13} \end{bmatrix}$$

(59)

1)

$$\begin{bmatrix} 2 & -1 \\ 7 & 3 \end{bmatrix}$$

2)

$$\begin{bmatrix} \frac{3}{13} & \frac{1}{13} \\ -\frac{7}{13} & \frac{2}{13} \end{bmatrix}$$

3)

$$\begin{bmatrix} 2 & -1 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} \frac{3}{13} & \frac{1}{13} \\ -\frac{7}{13} & \frac{2}{13} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(60)

a)

$$\vec{u} = -1\,\vec{w_1} + 4\vec{w_2} = \begin{bmatrix} +1 \\ -3 \end{bmatrix} + 4\begin{bmatrix} 7 \\ -2 \end{bmatrix} = \begin{bmatrix} 29 \\ -11 \end{bmatrix}$$

$$\vec{u}_{\{\vec{e_1}, \vec{e_2}\}} = \begin{bmatrix} 29 \\ -11 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 7 \\ 3 & -2 \end{bmatrix}$$

$$P \cdot [\vec{u}]_{\{\vec{w_1}, \vec{w_2}\}} = \begin{bmatrix} -1 & 7 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 29 \\ -11 \end{bmatrix}$$

~~61~~ 60 b)

$$\vec{s}_{[\vec{w}_1,\vec{w}_2]} = \cancel{\begin{bmatrix} -1 & 7 \\ 3 & -2 \end{bmatrix}^{-1}}$$

$$\vec{s}_{[\vec{w}_1,\vec{w}_2]} = \begin{bmatrix} -1 & 7 \\ 3 & -2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{-19} \begin{bmatrix} -2 & -7 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{19} \begin{bmatrix} 13 \\ -9 \end{bmatrix} = \begin{bmatrix} \frac{13}{19} \\ -\frac{9}{19} \end{bmatrix}$$

61 P: change of ~~basis~~ coordinates matrix from $\{\vec{w}_1, \vec{w}_2\}$ to $\{\vec{v}_1, \vec{v}_2\}$

$$\begin{bmatrix} 1 & -3 \\ -1 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 9 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 21 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} \\ \frac{1}{6} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 \\ -1 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 9 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 54 \\ 10 \end{bmatrix} = \begin{bmatrix} 9 \\ \frac{5}{3} \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{7}{2} & 9 \\ \frac{1}{6} & \frac{5}{3} \end{bmatrix}$$

(* updated 3/25)

$$\vec{x}_{\{\vec{v}_1,\vec{v}_2\}} = \begin{bmatrix} \frac{7}{2} & 9 \\ \frac{1}{6} & \frac{5}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} \cancel{6}\ 70 \\ \cancel{6}\ \cancel{36}\ \frac{36}{3} = 12 \end{bmatrix}$$

Solutions

(60)

a) $\vec{u} = -\vec{w}_1 + 4\vec{w}_2 = -\begin{bmatrix} -1 \\ 3 \end{bmatrix} + 4\begin{bmatrix} 7 \\ -2 \end{bmatrix} = \begin{bmatrix} 29 \\ -11 \end{bmatrix}$

$$P = \begin{bmatrix} -1 & 7 \\ 3 & -2 \end{bmatrix}$$

$$P \cdot [\vec{u}]_{\{\vec{w}_1, \vec{w}_2\}} = \begin{bmatrix} -1 & 7 \\ 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} (-1)\cdot(-1) + 4\cdot 7 \\ 3(-1) + (-2)\cdot 4 \end{bmatrix} = \begin{bmatrix} 29 \\ -11 \end{bmatrix}$$

b) $\vec{s} = x_1\vec{w}_1 + x_2\vec{w}_2$

$$\begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} x_1 + \begin{bmatrix} 7 \\ -2 \end{bmatrix} x_2$$

$$= \begin{bmatrix} -1 & 7 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 7 \\ 3 & -2 \end{bmatrix}^{-1} \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 7 \\ 3 & -2 \end{bmatrix} = \frac{1}{2-21} \cdot \begin{bmatrix} -2 & -7 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 2/19 & 7/19 \\ 3/19 & 1/19 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2/19 & 7/19 \\ 3/19 & 1/19 \end{bmatrix} \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} 13/19 \\ -9/19 \end{bmatrix}$$

$$\vec{s}_{\{\vec{w}_1, \vec{w}_2\}} = \begin{bmatrix} 13/19 \\ -9/19 \end{bmatrix}$$

60) cont. b)

$$P^{-1} \cdot [S]_{\{\vec{e_1}, \vec{e_2}\}} = \begin{bmatrix} 13/19 \\ -9/19 \end{bmatrix}$$

(61) P: Change of coordinate matrix from $\{\vec{w_1}, \vec{w_2}\}$ to $\{\vec{v_1}, \vec{v_2}\}$.

$$\vec{w_1} = x_1 \vec{v_1} + x_2 \vec{v_2}$$

$$\begin{bmatrix} 3 \\ -2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 9 \end{bmatrix} \qquad \cancel{x_1 - 3x_2}$$

$$= \begin{bmatrix} 1 & -3 \\ -1 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -1 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 9 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 21 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 1/6 \end{bmatrix}$$

$$\vec{w_1}_{\{\vec{v_1}, \vec{v_2}\}} = \begin{bmatrix} 7/2 \\ 1/6 \end{bmatrix}$$

$$\vec{w_2} = y_1 \vec{v_1} + y_2 \vec{v_2}$$

$$\begin{bmatrix} 4 \\ 6 \end{bmatrix} = y_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + y_2 \begin{bmatrix} -3 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -1 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 9 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 54 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 5/3 \end{bmatrix}$$

(61) Cont.

$$P = \begin{bmatrix} 7/2 & 9 \\ 1/6 & 5/3 \end{bmatrix}$$

$$\vec{x}_{\{\vec{v}_1, \vec{v}_2\}} = P \cdot \vec{x}_{\{\vec{w}_1, \vec{w}_2\}} = \begin{bmatrix} 7/2 & 9 \\ 1/6 & 5/3 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{6}\,70 \\ 1/3 + 35/3 \end{bmatrix} = \begin{bmatrix} 70 \\ 12 \end{bmatrix}$$

(62)

$$\cancel{f(\vec{w}_2)_{\{a\vec{w}_1, v}}$$

$$f(\vec{w}_2)_{\{\vec{v}_1, \vec{v}_2\}} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$\cancel{f}$ P: change-of-coordinate matrix from $\cancel{\{\vec{w}_1, \vec{w}_2\}}$ $\{\vec{v}_1, \vec{v}_2\}$ to $\{\vec{e}_1, \vec{e}_2\}$

$$f(\vec{w}_2)_{\{\vec{e}_1, \vec{e}_2\}} = P \cdot f(\vec{w}_2)_{\{\vec{v}_1, \vec{v}_2\}}$$

$$= \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

(b2) cont.)

$\cancel{f(\vec{w_1})}$. $\vec{w_1}_{\{\vec{v_1}, \vec{v_2}\}}$

Q: change-of-coordinate matrix from $\{\vec{e_1}, \vec{e_2}\}$ to $\{\vec{v_1}, \vec{v_2}\}$

$$Q = P^{-1} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$\cancel{f(\vec{w_1})_{e}} \quad \vec{w_1}_{\{\vec{v_1}, \vec{v_2}\}} = Q\, \vec{w_1}_{\{\vec{e_1}, \vec{e_2}\}}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$f(\vec{w_1})_{\cancel{\{\vec{e_1}, \vec{e_2}\}}\{\vec{v_1}, \vec{v_2}\}} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$f(\vec{w_1})_{\{\vec{e_1}, \vec{e_2}\}} = P\begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \end{bmatrix}$$

~~(63)~~ (66) $A\vec{v} = \vec{v}$

$$\begin{bmatrix} 5 & -3 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left(\begin{bmatrix} 5 & -3 \\ -4 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & -3 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$4x_1 - 3x_2 = 0$$

$$x_1 = \frac{3}{4}x_2$$

$x_2 = 1 \qquad x_1 = \frac{3}{4}$

$$\vec{v} = \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$$

(70) ~~(* updated 4/1)~~

P: change-of-~~co~~ coordinate matrix from $\{\vec{w_1}, \vec{w_2}\}$ to $\{\vec{v_1}, \vec{v_2}\}$

$$P = \begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix}$$

Q - - - — — - —. $\{\vec{v_1}, \vec{v_2}\}$ to $\{\vec{w_1}, \vec{w_2}\}$

$$Q = P^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -3 & -2 \end{bmatrix}$$

$$P^{-1}\begin{bmatrix} 7 & -1 \\ 0 & 2 \end{bmatrix} P = \begin{bmatrix} -1 & -1 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} 7 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & -1 \\ -21 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -17 & 6 \\ -39 & 20 \end{bmatrix}$$

(* updated 4/1)

Solutions

(67) $B\vec{v} = 2\vec{v}$

$$\begin{bmatrix} 3 & -4 & -3 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\left(\begin{bmatrix} 3 & -4 & -3 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & -4 & -3 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\xrightarrow{R_2 + R_3 \to R_3} \begin{bmatrix} 1 & -4 & -3 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_1 + x_3 = 0$

$x_1 = -x_3$

$x_1 - 4x_2 - 3x_3 = 0$

$-x_3 - 4x_2 - 3x_3 = 0$

$x_2 = -x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$x_3 = 1$. $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ one eigenvector

69.

a) $A - \lambda I = \begin{bmatrix} 11-\lambda & -4 \\ 30 & -11-\lambda \end{bmatrix}$

$\det(A - \lambda I) = (11-\lambda)(-11-\lambda) - (-120)$

$= -121 + \lambda^2 + 120 = \lambda^2 - 1 \boxed{= 0}$

$\lambda = \pm 1$

two eigenvalues

b) $B - \lambda I = \begin{bmatrix} 2-\lambda & \\ & 2-\lambda \end{bmatrix}$

$\det(B - \lambda I) = (2-\lambda)^2 = 0$

$\lambda = 2$

one eigenvalue

c) $C - \lambda I = \begin{bmatrix} -7-\lambda & 4 \\ -25 & 11-\lambda \end{bmatrix}$

$\det(C - \lambda I) = (-7-\lambda)(11-\lambda) - (-100)$

$= -77 - 11\lambda + 7\lambda + \lambda^2 + 100$

$= \lambda^2 - 4\lambda + 23$ No solution

No eigenvalues

71

~~$A-\lambda I$~~ $A = \begin{bmatrix} 3 & \\ & 3 \end{bmatrix}$: eigenvalue $\lambda = 3$

a)

$$(A - \lambda I)\vec{v} = 0$$

$$\left(\begin{bmatrix} 3 & \\ & 3 \end{bmatrix} - \begin{bmatrix} 3 & \\ & 3 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 = 1 \qquad x_2 = 0 \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$x_1 = 0 \qquad x_2 = 1 \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

dimension of solution space

# variables	-	# equations	
2		0	= 2

eigenspace of eigenvalue 3:

span $\left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$

b)

$$B - \lambda I = \begin{bmatrix} 8-\lambda & -4 \\ 9 & -4-\lambda \end{bmatrix}$$

$$\det(B - \lambda I) = (8-\lambda)(-4-\lambda) - (-36)$$

$$= -32 + 4\lambda - 8\lambda + \lambda^2 + 36$$

$$= \lambda^2 - 4\lambda + 4$$

$$= (\lambda - 2)^2 = 0 \qquad \lambda = 2$$

$$(B-2I)\vec{v} = 0$$

$$\begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$R_1 \cdot (-1.5) + R_2 \sim R_2$$

$$\begin{bmatrix} 6 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$6x_1 - 4x_2 = 0$$

$$x_1 = \frac{2}{3} x_2$$

$$\vec{v} = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$$

eigenspace of eigenvalue 2:

$$\text{span} \left\langle \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} \right\rangle$$

72

Answer.

$$M = \begin{bmatrix} 0.7 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.3 \\ 0.1 & 0.3 & 0.5 \end{bmatrix}$$

$\vec{v}$: an eigenvector of eigenvalue 1

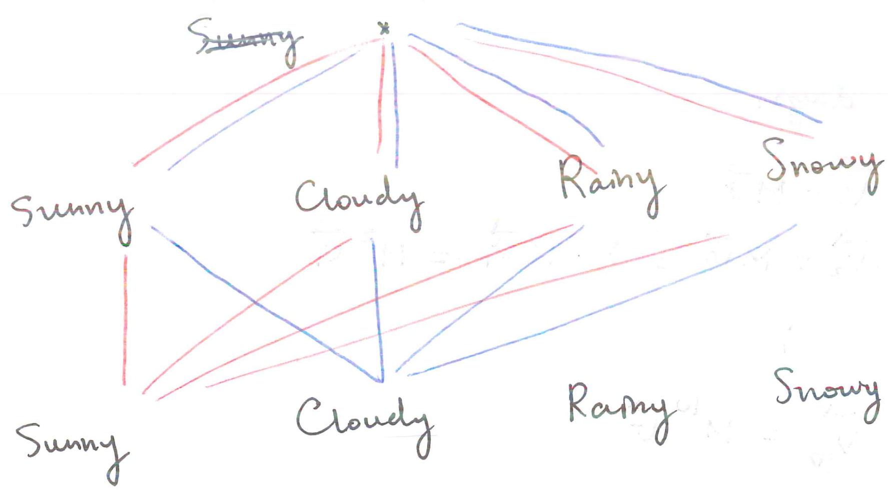
Explanation

$$\begin{cases} 0.7x_1 + 0.4x_2 + 0.2x_3 = x_1 \\ 0.2x_1 + 0.4x_2 + 0.3x_3 = x_2 \\ 0.1x_1 + 0.3x_2 + 0.5x_3 = x_3 \end{cases}$$

$$\begin{bmatrix} 0.7 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.3 \\ 0.1 & 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$M\vec{v} = \vec{v}$$

73



Sunny:

$$x_1 = 0.1 \times 0.5 + 0.2 \times \overset{0.3}{\cancel{0.1}} + 0.3 \times \overset{0.1}{\cancel{0.2}} + 0.4 \times 0.2$$

$$= \begin{bmatrix} 0.5 & 0.3 & 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \end{bmatrix}$$

(73) Cont.

Cloudy

$$x_2 = \begin{bmatrix} 0.1 & 0.4 & 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \end{bmatrix}$$

Rainy

$$x_3 = \begin{bmatrix} 0.2 & 0.1 & 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \end{bmatrix}$$

Snowy

$$x_4 = \begin{bmatrix} 0.2 & 0.2 & 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \end{bmatrix}$$

State vector

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.3 & 0.1 & 0.2 \\ 0.1 & 0.4 & 0.3 & 0.1 \\ 0.2 & 0.1 & 0.3 & 0.1 \\ 0.2 & 0.2 & 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \end{bmatrix}$$

After 10 days:

$$\vec{v}_1 = M\vec{v}$$

$$\vec{v}_2 = M\vec{v}_1 = M \cdot M\vec{v}_1 = M^2 \vec{v}_1$$

$$\vdots$$

$$\vec{v}_{10} = M^{10} \vec{v}$$

$M\vec{w} = \vec{w}$. Stable-state vector

(76)

$$M\vec{v} = \vec{v}$$

$$\begin{bmatrix} 0.3 & 0.9 \\ 0.7 & 0.1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = I \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left(\begin{bmatrix} 0.3 & 0.9 \\ 0.7 & 0.1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -0.7 & 0.9 \\ 0.7 & -0.9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -0.7 & 0.9 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$0.7x_1 = 0.9x_2$$

$$\begin{cases} x_1 = \frac{9}{7}x_2 \\ x_1 + x_2 = 1 \end{cases}$$

$$\frac{9}{7}x_2 + x_2 = 1$$

$$x_2 = \frac{7}{16}$$

$$x_1 = \frac{9}{7} \cdot \frac{7}{16} = \frac{9}{16}$$

$$\vec{v} = \begin{bmatrix} \frac{9}{16} \\ \frac{7}{16} \end{bmatrix}$$

(74)

$$P = \begin{bmatrix} 2 & 7 \\ 4 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & \\ & -2 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 2 & 7 \\ 4 & 1 \end{bmatrix}^{-1}$$

$$A^{100} = (PAP^{-1})^{100} = \underbrace{PA\cancel{P^{-1}} \cdot \cancel{P}A\cancel{P^{-1}} \cdots \cancel{P}AP^{-1}}_{100 \text{ times}}$$

$$= P \cdot D^{100} P^{-1}$$

$$= \begin{bmatrix} 2 & 7 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & \\ & -2 \end{bmatrix}^{100} \begin{bmatrix} 2 & 7 \\ 4 & 1 \end{bmatrix}^{-1}$$

(74) cont).

$$A^{100} = \begin{bmatrix} 2 & 7 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 2^{100} \end{bmatrix} \cdot \frac{1}{-26} \begin{bmatrix} 1 & -7 \\ -4 & 2 \end{bmatrix}$$

$$= -\frac{1}{26} \begin{bmatrix} 2 & 7 \cdot 2^{100} \\ 4 & 2^{100} \end{bmatrix} \begin{bmatrix} 1 & -7 \\ -4 & 2 \end{bmatrix}$$

$$= -\frac{1}{26} \begin{bmatrix} 2 - 28 \cdot 2^{100} & -14 + 14 \cdot 2^{100} \\ 4 - 4 \cdot 2^{100} & -28 + 2 \cdot 2^{100} \end{bmatrix}$$

(* updated 4/4)